

On deformations of $AdS_n \times S^n$ supercosets

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Abstract

We study the deformed $AdS_5 \times S^5$ supercoset model of arXiv:1309.5850 which depends on one parameter \varkappa and has classical quantum group symmetry. We confirm the conjecture that in the “maximal” deformation limit, $\varkappa \rightarrow \infty$, this model is T-dual to “flipped” double Wick rotation of the target space $AdS_5 \times S^5$, i.e. $dS_5 \times H^5$ space supported by an imaginary 5-form flux. In the imaginary deformation limit, $\varkappa \rightarrow i$, the corresponding target space metric is of a pp-wave type and thus the resulting light-cone gauge S-matrix becomes relativistically invariant. Omitting non-unitary contributions of imaginary WZ terms, we find that this tree-level S-matrix is equivalent to that of the generalized sine-Gordon model representing the Pohlmeyer reduction of the undeformed $AdS_5 \times S^5$ superstring model. We also study in some detail similar deformations of the $AdS_3 \times S^3$ and $AdS_2 \times S^2$ supercosets. The bosonic part of the deformed $AdS_3 \times S^3$ model happens to be equivalent to the symmetric case of the sum of the Fateev integrable deformation of the $SL(2)$ and $SU(2)$ principal chiral models, while in the $AdS_2 \times S^2$ case the role of the Fateev model is played by the 2d “sausage” model. The $\varkappa = i$ limits are again directly related to the Pohlmeyer reductions of the corresponding $AdS_n \times S^n$ supercosets: (2,2) super sine-Gordon model and its complex sine-Gordon analog. We also discuss possible deformations of $AdS_3 \times S^3$ with more than one parameter.

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Contents

| | | |
|----------|---|-----------|
| 1 | Introduction | 2 |
| 2 | Deformed $AdS_5 \times S^5$ model and its limits | 5 |
| 2.1 | $\varkappa = \infty$ limit | 6 |
| 2.2 | $\varkappa = i$ limit | 8 |
| 2.3 | Consistent truncations to low-dimensional models | 11 |
| 3 | Deformed $AdS_3 \times S^3$ model | 12 |
| 3.1 | \varkappa -deformed S^3 as the “symmetric” case of the Fateev model | 13 |
| 3.2 | $\varkappa = i$ limit: equivalence to the PR model for the $AdS_3 \times S^3$ superstring | 15 |
| 4 | Deformed $AdS_2 \times S^2$ model | 19 |
| 4.1 | Deformed supercoset Lagrangian | 19 |
| 4.2 | Special case of $\varkappa = i$ | 21 |
| 5 | Concluding remarks | 22 |
| A | Equivalence of 2-parameter $SU(2)$ Yang-Baxter sigma model to Fateev model | 23 |
| B | 4-parameter integrable 3d model with a WZ term | 25 |
| C | The $\varkappa = i$ action from the $AdS_2 \times S^2$ supercoset | 28 |

1 Introduction

The integrability of the $AdS_5 \times S^5$ superstring theory provides an important tool for finding its spectrum [1]. Given an integrable sigma model one may construct closely related integrable models by applying T-duality transformations (see, e.g., [2, 3, 4, 5, 6]). Recently, a novel example of a one-parameter integrable deformation of the $AdS_5 \times S^5$ supercoset model, not related to T-duality was found in [7] (following earlier constructions of [8, 9, 10]). In [11] the coordinate form of the bosonic part of the corresponding string action was worked out and the background string metric and the NS-NS 2-form were explicitly determined. This 10d background has the $SO(2, 4) \times SO(6)$ symmetry of $AdS_5 \times S^5$ broken to its Cartan subgroup $[U(1)]^6$ and thus its dual gauge theory interpretation is not immediately clear.

The deformed string model [7] is parametrized by the string tension $T_0 \equiv g = \frac{\sqrt{\lambda}}{2\pi}$ and a real deformation parameter $\eta \in [0, 1)$. It is useful to also introduce related parameters $\varkappa \in [0, \infty)$ and q as in [11]

$$\varkappa = \frac{2\eta}{1 - \eta^2} , \quad q = e^{-\nu/g} , \quad \nu = \frac{2\eta}{1 + \eta^2} = \frac{\varkappa}{\sqrt{1 + \varkappa^2}} . \quad (1.1)$$

An interesting feature of the model of [7] is its classical q -deformed symmetry, suggesting that it is more symmetric than is apparent from its Lagrangian description.¹ Remarkably, the corresponding tree-level light-cone (bosonic) S-matrix matched [11] the S-matrix with q -deformed centrally-extended $[psu(2|2)]^2$ symmetry [14, 15, 16] with real deformation parameter q .

This leaves many open questions. In particular, it is not clear whether this deformation should have an interesting target space interpretation or if it is just a member of a universality class of models with effectively equivalent classical integrable structure, but related to the original undeformed one by non-local transformations making the resulting quantum theories inequivalent. Another interesting question is about the existence and properties of a gauge theory dual to string theory in the deformed geometry.

Our aim here will be to explore this deformed model by studying its simple limits and low-dimensional analogs. In particular, we shall consider in detail the following two formal limits:

- (i) $\eta = 1$ or $\varkappa = \infty$ ($q = e^{-1/g}$) and (ii) $\eta = i$ or $\varkappa = i$ ($q = e^{i\infty/g}$).

It turns out that in the first “maximal deformation” limit the deformed 10d metric becomes closely related (T-dual) to a flipped “double Wick rotation” of the $AdS_5 \times S^5$ space – $dS_5 \times H^5$, where dS_5 is the de Sitter space (whose euclidean continuation is S^5) and H^5 is the 5d hyperboloid (which is the euclidean continuation of AdS_5). This proves the conjecture of [7] that the deformation effectively interpolates between $AdS_5 \times S^5$ and $dS_5 \times H^5$ spaces. $dS_5 \times H^5$ is a formal solution of type IIB supergravity supported by an imaginary self-dual 5-form flux [17] implying that the corresponding world-sheet theory is non-unitary.

In the second “imaginary deformation” limit (combined with a particular rescaling of coordinates) the 10d metric becomes that of a pp-wave background with a curved transverse part. The corresponding light-cone gauge string action takes a form reminiscent of the action of the Pohlmeyer reduced (PR) theory for the $AdS_5 \times S^5$ superstring² [18, 19, 20] but with additional imaginary B -field (WZ) terms, implying that unitarity is broken. The resulting light-cone gauge S-matrix is then *relativistically invariant* and, ignoring the imaginary WZ term contribution, happens to be the same as the tree-level PR S-matrix found in [21, 22].

We shall also study the direct 6d and 4d analogs of the deformed $AdS_5 \times S^5$ model, which may be interpreted as deformations of the $AdS_3 \times S^3$ [23] and $AdS_2 \times S^2$ [24] supercosets. The corresponding metrics are direct sums of deformed AdS_n and deformed S^n metrics and are simply given by truncations of the corresponding parts of the deformed 10d metric of [11]. The integrability of the resulting 6d and 4d classical string models is inherited from the integrability of 10d model.

As we shall explain below, the corresponding bosonic integrable models were identified before in a different guise: the one-parameter \varkappa -deformation of the S^3 metric (corresponding to a deformation of the $SO(4)/SO(3)$ coset following [8]) is a special “left-right symmetric” case of the Fateev 2-parameter deformation of the $SU(2)$ principal chiral model [25], the classical integrability of which was proved in [26]. Similarly, the \varkappa -deformation of the S^2 metric (found also as the $SU(2)/U(1)$ coset deformation in [8]) is the same as the 2d “sausage” model of [27],

¹Similar models with classical q -deformed symmetry were discussed in [12, 13].

²The Pohlmeyer reduction is based on solving the string Virasoro conditions in terms of new “current” variables related through derivatives (i.e. non-locally) to the original string coordinates and then writing down the action for the new unconstrained variables.

for which the classical Lax pair was given in [26].

We shall show that the general 2-parameter Fateev model [25] is the same as the $SU(2)$ case of the 2-parameter family of classically integrable “bi-Yang-Baxter” sigma models constructed in [10, 28].³ This suggests the existence of a two-parameter deformation of the $AdS_3 \times S^3$ supercoset model with the bosonic part being given by the sum of the $SU(2)$ Fateev model and its $SL(2, R)$ analog. Furthermore, the Fateev model admits an integrable extension [26] to the presence of a WZ term or non-zero B -field coupling, implying that it might be possible to construct a 3-parameter deformation of the $AdS_3 \times S^3$ supercoset [30] with non-zero NS-NS B -field coupling (containing as a special case the $SL(2, R) \times SU(2)$ WZW model).

For these low-dimensional $AdS_3 \times S^3$ and $AdS_2 \times S^2$ models one may also study the special $\varkappa = \infty$ and $\varkappa = i$ limits. In particular, in the $\varkappa = i$ limit the resulting pp-wave model turns out to be closely related to the one in [31, 32, 33]. After completing these pp-wave 4d and 6d metrics to supergravity solutions we will find that in light-cone gauge they reduce, in the 4d case, to the $(2, 2)$ supersymmetric sine-Gordon model which is equivalent to the PR model for the $AdS_2 \times S^2$ superstring [18], and, in the 6d case, to a fermionic extension of the sum of the complex sine-Gordon and complex sinh-Gordon models which is equivalent to the PR model [20] for the $AdS_3 \times S^3$ superstring and has hidden $(4, 4)$ supersymmetry. In the $AdS_2 \times S^2$ case, we will also construct explicitly the quadratic fermion terms in the $AdS_2 \times S^2$ analog of the deformed supercoset action of [7] for $\varkappa = i$ and show that it also reproduces the PR model for the $AdS_2 \times S^2$ superstring [18]. A similar analysis should be possible for the $AdS_3 \times S^3$ and $AdS_5 \times S^5$ cases as well.

We shall start in section 2 with a review of the 10d \varkappa -dependent metric and B -field background corresponding [11] to the deformed $AdS_5 \times S^5$ model of [7] and then consider the special limits of $\varkappa = \infty$ and $\varkappa = i$ and low-dimensional truncations.

The deformed $AdS_3 \times S^3$ case will be discussed in detail in section 3, where we explain the equivalence of the \varkappa -deformed S^3 metric to the symmetric case of the Fateev model and discuss the relation between the $\varkappa = i$ limit of the deformed $AdS_3 \times S^3$ background and the Pohlmeyer reduced model for the original undeformed $AdS_3 \times S^3$ superstring theory.

Section 4 is devoted to the $AdS_2 \times S^2$ case. We shall start with the deformed $AdS_2 \times S^2$ supercoset Lagrangian constructed following [7] and show that its bosonic part corresponds to the 4d truncation of the 10d \varkappa -deformed metric. We shall then consider the $\varkappa = i$ case and show its equivalence in the light-cone gauge to the PR model for the undeformed $AdS_2 \times S^2$ supercoset. We shall also demonstrate that the deformed sigma model is one-loop UV finite when expanded near a BMN-type geodesic.

In Appendix A we demonstrate the equivalence between the Fateev model [25] and the 2-parameter $SU(2)$ bi-Yang-Baxter sigma model of [10, 28]. Appendix B presents a review of the 4-parameter Lukyanov sigma model [26], which generalizes the Fateev model introducing a B -field coupling. Appendix C contains the details of the construction of the deformed supercoset action for $AdS_2 \times S^2$ at $\varkappa = i$ and the demonstration of its equivalence to the Pohlmeyer

³The “diagonal” ($\alpha = \beta$) limit of the 2-parameter $SU(2)$ YB model of [28] is the same as the $SO(4)/SO(3)$ coset deformation of [8] or the “symmetric” one-parameter case of the Fateev model. The one-parameter ($\beta = 0$) case is the same as the original Yang-Baxter model of [9, 10] which, in the $SU(2)$ case, is the squashed S^3 model known to be integrable since [29].

reduced model for the undeformed $AdS_2 \times S^2$ superstring.

2 Deformed $AdS_5 \times S^5$ model and its limits

The deformed $AdS_5 \times S^5$ string action may be written as [7, 11]

$$S = \frac{1}{2}\hat{T} \int d^2\sigma (L_G + L_B + L_{\text{ferm}}) , \quad \hat{T} = g(1 + \varkappa^2)^{1/2} , \quad (2.1)$$

where \hat{T} is the effective string tension⁴ L_G is the string metric G part and L_B is the 2-form B or WZ part. The fermionic terms should contain couplings to $e^\Phi F_k$ ($k = 1, 3, 5$) where Φ is the dilaton (which is non-constant for generic \varkappa) and F_k are RR fluxes. Φ and F_k (which are presently unknown) should supplement G and B to give a type IIB supergravity solution to ensure conformal invariance of the model as suggested by the fermionic kappa-symmetry of the deformed action [7].

Explicitly, the deformed analog of the AdS_5 metric is [11]

$$ds_{A_5}^2 = -h(\rho)dt^2 + f(\rho)d\rho^2 + \rho^2 [v(\rho, \zeta)(d\zeta^2 + \cos^2 \zeta d\psi_1^2) + \sin^2 \zeta d\psi_2^2] , \quad (2.2)$$

$$h = \frac{1 + \rho^2}{1 - \varkappa^2 \rho^2} , \quad f = \frac{1}{(1 + \rho^2)(1 - \varkappa^2 \rho^2)} , \quad v = \frac{1}{1 + \varkappa^2 \rho^4 \sin^2 \zeta} . \quad (2.3)$$

For $\varkappa = 0$ this is the standard global AdS_5 metric with embedding coordinates $X_0 + iX_5 = \sqrt{1 + \rho^2}e^{it}$, $X_1 + iX_2 = \rho \cos \zeta e^{i\psi_1}$, $X_3 + iX_4 = \rho \sin \zeta e^{i\psi_2}$. The deformed S^5 metric is found by a simple analytic continuation $\rho \rightarrow ir$ and reversing the overall sign of the metric:

$$ds_{S_5}^2 = \tilde{h}(r)d\varphi^2 + \tilde{f}(r)dr^2 + r^2 [\tilde{v}(r, \theta)(d\theta^2 + \cos^2 \theta d\phi_1^2) + \sin^2 \theta d\phi_2^2] , \quad (2.4)$$

$$\tilde{h} = \frac{1 - r^2}{1 + \varkappa^2 r^2} , \quad \tilde{f} = \frac{1}{(1 - r^2)(1 + \varkappa^2 r^2)} , \quad \tilde{v} = \frac{1}{1 + \varkappa^2 r^4 \sin^2 \theta} . \quad (2.5)$$

The non-zero B_{mn} components in the two subspaces are

$$B_{\psi_1 \zeta} = \frac{1}{2}\varkappa \rho^4 \sin 2\zeta v(\rho, \zeta) , \quad B_{\phi_1 \theta} = -\frac{1}{2}\varkappa r^4 \sin 2\theta \tilde{v}(r, \theta) . \quad (2.6)$$

The deformed metrics (2.2), (2.4) have only the $[U(1)]^3$ Cartan subgroups of the original $SO(2, 4)$ and $SO(6)$ as their surviving symmetry. The deformed target space background should not have (for generic value of \varkappa) any manifest supersymmetry but the string model (2.1) of [7] should have hidden symmetries due to its integrability.

Assuming that the above 10d metric and B -field background can indeed be completed to a full type II supergravity solution⁵ the corresponding dilaton should satisfy the following equation

$$R + 4\nabla^2 \Phi - 4(\nabla_m \Phi)^2 - \frac{1}{12}H_{mnk}^2 = 0, \quad \text{i.e.} \quad \nabla^2 e^{-\Phi} + \frac{1}{4}(R - \frac{1}{12}H_{mnk}^2)e^{-\Phi} = 0 . \quad (2.7)$$

⁴We use the definition of the tension from [11]; the choice in [7] was $\hat{T} = g(1 + \varkappa^2)^{3/2}$.

⁵This is a non-trivial assumption: it does not seem likely that a generic NS-NS (G, B) background may be completed by a dilaton and RR fluxes to a type II supergravity solution. In the present case this is expected provided the deformed supercoset model of [7] does indeed have an interpretation as a GS action in a type IIB supergravity background [34]. This is supported, in particular, by its kappa-symmetry [7] and the special limits of $\varkappa = \infty$ and $\varkappa = i$ when this is the case as discussed below.

This equation does not appear to have a simple rational solution for the above background, suggesting that the RR fluxes F_k should also have a complicated form, such that $e^\Phi F_k$ is rational. This is required to match the structure of the fermionic terms in (2.1) that should have a rational dependence on coordinates, as implied by the construction of [7] for the coordinate parametrization used in [11].

The deformed string metric (2.2) has a curvature singularity at $\rho_* = 1/\varkappa$; for larger values the radial coordinate ρ becomes time-like, suggesting that strings are confined to the region $0 \leq \rho \leq 1/\varkappa$. Thus for $\varkappa \neq 0$ the metric (2.2) no longer has a boundary (which reappears at $\rho = \infty$ if $\varkappa = 0$). It is an open question if/how string theory resolves this singularity. It would also be interesting to understand in detail whether string theory in the deformed geometry (2.3) supplemented by the required fluxes and dilaton has a gauge theory dual. In the absence of non-abelian isometries it should not have conformal or even Lorentz symmetry (and of course no manifest supersymmetry).⁶

2.1 $\varkappa = \infty$ limit

Let us now consider the “maximal deformation” limit, $\varkappa \rightarrow \infty$ (or $\eta = 1$), in the string action (2.1). If we formally take this limit in the metric (2.2), (2.3) we get

$$ds_{A_5, \varkappa \rightarrow \infty}^2 = \frac{1}{\varkappa^2} \bar{ds}_{A_5}^2, \quad \bar{ds}_{A_5}^2 = (1 + \bar{\rho}^2) dt^2 - (1 + \bar{\rho}^2)^{-1} d\bar{\rho}^2 + \bar{\rho}^2 ds_3^2, \quad (2.8)$$

$$ds_3^2 = \frac{d\zeta^2}{\sin^2 \zeta} + \cot^2 \zeta d\psi_1^2 + \bar{\rho}^{-4} \sin^2 \zeta d\bar{\psi}_2^2, \quad (2.9)$$

$$\bar{\rho} \equiv \rho^{-1}, \quad \bar{\psi}_2 \equiv \varkappa \psi_2. \quad (2.10)$$

Here we have redefined ρ and ψ_2 (so that $\bar{\psi}_2$ is non-compact for $\varkappa \rightarrow \infty$). The corresponding B term in (2.1), (2.6) becomes a total derivative in this limit and can be ignored. Performing a formal T-duality along the $\bar{\psi}_2$ direction and changing the coordinate $\zeta \rightarrow y$ we find that the T-dual metric becomes ($y \equiv \ln \tan \frac{\zeta}{2}$)⁷

$$\tilde{\bar{ds}}_{A_5}^2 = ds_{dS_5}^2 = -(1 + \bar{\rho}^2)^{-1} d\bar{\rho}^2 + (1 + \bar{\rho}^2) dt^2 + \bar{\rho}^2 (dy^2 + \sinh^2 y d\psi_1^2 + \cosh^2 y d\tilde{\psi}_2^2). \quad (2.11)$$

This metric is the same as the metric of the *de Sitter* (dS_5) space with $\bar{\rho}$ now playing the role of the time coordinate.⁸ Thus, while at $\varkappa = 0$ the metric (2.2) is that of the negative-curvature AdS_5 space, at $\varkappa = \infty$ it is T-dual to the positive-curvature dS_5 metric.

Similarly, the $\varkappa \rightarrow \infty$ limit of the deformed S^5 metric (2.4) becomes (after the analogous coordinate transformations, $\bar{r} = r^{-1}$, etc., and T-duality) the metric of the negative-curvature

⁶A singularity in the Einstein frame metric would suggest that the UV limit of this theory is in some sense not well-defined perhaps due to deformation by an irrelevant operator. A similar conclusion may be reached by analysing the deformation in coordinates in which the metric reduces to that of the $AdS_5 \times S^5$ in the Poincaré patch as $\varkappa \rightarrow 0$.

⁷We assume that the dual coordinate $\tilde{\psi}$ is rescaled by $\hat{T}\varkappa^{-2}$, where \hat{T} is the string tension in (2.1).

⁸Setting $\bar{\rho} = \sinh \xi$ we get $ds_{dS_5}^2 = -d\xi^2 + \cosh^2 \xi dt^2 + \sinh^2 \xi (dy^2 + \sinh^2 y d\psi_1^2 + \cosh^2 y d\tilde{\psi}_2^2)$. The scalar curvature of this metric is $R = 20$. One can introduce global coordinates in $R^{1,5}$ such that this metric becomes that of the positive-curvature surface $-X_0^2 + X_1^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 = 1$.

euclidean AdS_5 space or the hyperboloid H^5 ($x \equiv \ln \tan \frac{\theta}{2}$):⁹

$$\tilde{ds}_{S_5}^2 = ds_{H^5}^2 = (\bar{r}^2 - 1)^{-1} d\bar{r}^2 + (\bar{r}^2 - 1) d\varphi^2 + \bar{r}^2 (dx^2 + \sinh^2 x d\phi_1^2 + \cosh^2 x d\tilde{\phi}_2^2) . \quad (2.12)$$

We conclude that the \varkappa -deformation interpolates between $AdS_5 \times S^5$ at $\varkappa = 0$ and the T-dual of $dS_5 \times H^5$ at $\varkappa = \infty$. This effectively confirms the conjecture made in [7]¹⁰ which was motivated by a similar interpolation between the $SU(2)/U(1)$ and $SU(1,1)/U(1)$ cosets observed in [8].

It is interesting to note that $dS_5 \times H^5$ is also a double Wick rotation of $AdS_5 \times S^5$ combined with a Z_2 interchange of the factors (the euclidean rotation of AdS_5 is H^5 and the Minkowski version of S^5 is dS_5). For that reason this space also solves the (complexified) type IIB supergravity equations if it is supported by an *imaginary* self-dual 5-form flux (the 5+5 Ricci tensor blocks should change signs as compared to the $AdS_5 \times S^5$ case, which is supported by a real self-dual 5-form flux).¹¹

Reversing the T-duality transformations along ψ_2 and ϕ_2 we get a type IIB supergravity solution (with the metric in (2.8), (2.9) and its S^5 counterpart) supported by an imaginary constant self-dual F_5 flux and the following dilaton field (originating from the T-duality transformations) $\Phi = \Phi_A + \Phi_S$

$$(\Phi_A)_{\varkappa=\infty} = -\ln(\bar{\rho}^2 \cosh y) = \ln(\rho^2 \sin \zeta) , \quad (\Phi_S)_{\varkappa=\infty} = -\ln(\bar{r}^2 \cosh x) = \ln(r^2 \sin \theta) . \quad (2.13)$$

The fact that the $\varkappa = \infty$ limit of the deformed background is a formal solution of type IIB supergravity (with an imaginary F_5 flux) verifies that the corresponding limit of the deformed superstring action [7] should be describing a 2d conformal theory. However, the presence of an imaginary $e^\Phi F_5$ coupling in the fermionic part of the string action suggests that the resulting world-sheet theory is likely to be *non-unitary*.

This non-unitarity is probably related to a special nature of the limit $\varkappa = \infty$ (or $\eta = 1$): in this case the quantum deformation parameter q in (1.1) approaches unity in the perturbative string limit, $g \rightarrow \infty$.¹²

Interestingly, such a limit taken in the interpolating S-matrix [14, 16] with real q formally

⁹This metric of H^5 is written in the hyperbolic slicing; its scalar curvature is $R = -20$. Indeed, one can introduce global coordinates in $R^{1,5}$ such that this metric becomes the metric of the negative-curvature surface $-X_5^2 + X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = -a^2$. Similarly, the de Sitter space corresponds to the positive-curvature surface $-X_0^2 + X_5^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = a^2$ (we set the radius $a = 1$ in above expressions). One may choose the static coordinates as $X_0 = \sqrt{a^2 - \varrho^2} \sinh(t/a)$, $X_5 = \sqrt{a^2 - \varrho^2} \cosh(t/a)$, $X_k = \varrho n_k$ (where n_k are the 3-sphere coordinates, $n_k n_k = 1$) in which the dS_5 metric becomes $ds_{dS_5}^2 = -(1 - \varrho^2/a^2) dt^2 + (1 - \varrho^2/a^2)^{-1} d\varrho^2 + \varrho^2 d\Omega_3^2$. The analogous metric for H^5 is $ds_{H^5}^2 = (1 + \xi^2/a^2) d\phi^2 + (1 + \xi^2/a^2)^{-1} d\xi^2 + \xi^2 d\Omega_3^2$.

¹⁰T-duality was not mentioned in [7] but at the level of the first-order formalism used in [8] it may be viewed as a kind of canonical transformation (which is non-local in terms of the original coordinate fields).

¹¹Such a solution of a non-unitary analytic continuation of type IIB supergravity was discussed earlier in [17].

¹²Since for $\varkappa \rightarrow \infty$ the metric in (2.8) scales as \varkappa^{-2} and the tension in (2.1) goes as $\hat{T} \sim g\varkappa$ to get a finite string action in this limit one would need to rescale g by \varkappa . This will make q in (1.1) go to 1. Alternatively, we may keep g fixed and rescale the string coordinates to cancel the overall factor of \varkappa^{-1} . In this case (in full analogy with the $AdS_5 \times S^5$ case [36, 37], using the static coordinates of footnote 9) we will end up with a pp-wave limit of the $dS_5 \times H^5$ background, $ds^2 = 4dx^+ dx^- + (x_r^2 + y_m^2) dx^{+2} + dx_r dx_r + dy_m dy_m$. Then the 4+4 massive bosonic fluctuations found in light-cone gauge will be tachyonic. Similarly, the fermionic mass terms (which will be imaginary due to the imaginary F_5 flux) will also correspond to non-unitary tachyonic modes.

corresponds to the mirror theory S-matrix (cf. [38])¹³ and in this context it is not clear why non-unitarity should appear for real q . One possible resolution of this puzzle is to consider the light-cone gauge-fixed string in the $dS_5 \times H^5$ background in static coordinates and formally interchange the world-sheet coordinates or, equivalently, the world-sheet energy and momentum. The dispersion relation of the tachyonic modes discussed in footnote 12 then becomes the usual massive one. Assuming this prescription also extends to the interaction terms, the light-cone gauge-fixed mirror theory should then be equivalent to the light-cone gauge-fixing of the string in the $dS_5 \times H^5$ background with the world-sheet coordinates (and the corresponding conserved charges) formally interchanged.

There is also a more general perspective on this (non)unitarity issue. The deformed AdS_5 metric (2.2) contains factors of $1 - \varkappa^2 \rho^2$ implying that ρ is formally restricted to the interval $0 \leq \rho < \varkappa^{-1}$ ($\rho = \varkappa^{-1}$ is the curvature singularity). Continuing ρ beyond \varkappa^{-1} implies that ρ becomes time-like, while t becomes space-like. Also, the RR fluxes may contain factors of $\sqrt{1 - \varkappa^2 \rho^2}$ and so they may become imaginary for $\rho > \varkappa^{-1}$. This may be an indication that the “unphysical” region $\varkappa^{-1} \leq \rho < \infty$ is describing a non-unitary world-sheet theory. The $\varkappa = \infty$ limit discussed above corresponds to the case when the “physical” region $0 \leq \rho < \varkappa^{-1}$ shrinks to a point while the “unphysical” one extends to the whole half-line. Starting with the unitary light-cone gauge S-matrix found as in [11] in the “physical” $0 \leq \rho < \varkappa^{-1}$ region and taking the formal limit $\varkappa \rightarrow \infty$ corresponds effectively to switching to the S-matrix in the “unphysical” region. It should be noted that the light-cone gauge fixing and $\varkappa \rightarrow \infty$ limit need not commute. Indeed, the original BMN geodesic in the $0 \leq \rho < \varkappa^{-1}$ region may become complex in the “unphysical” region, while the non-unitarity of the $dS_5 \times H^5$ S-matrix refers to the expansion near a different vacuum – the real BMN type geodesic in static coordinates. This may be a resolution of the tension with unitarity of the mirror S-matrix.

Finally, let us note also that this intriguing relation of $AdS_5 \times S^5$ to its double Wick rotation $dS_5 \times H^5$ via the η -deformation is potentially hinting at a more universal description in terms of complexification of the underlying (super)group or in terms of its “double” (cf. [9]).

2.2 $\varkappa = i$ limit

Even though the model of [7] is defined for real η , let us study the formal limit of $\eta \rightarrow i$ or $\varkappa \rightarrow i$ as it has some interesting features and, in particular, establishes a connection to the Pohlmeyer reduced model for the original ($\varkappa = 0$) $AdS_5 \times S^5$ superstring.

Directly setting $\varkappa = i$ in the metrics (2.2) and (2.4) we observe that t and φ directions decouple

$$ds_{A_5, \varkappa=i}^2 = -dt^2 + ds_{A\perp}^2, \quad ds_{A\perp}^2 = \frac{d\rho^2}{(1 + \rho^2)^2} + \rho^2 \left[\frac{d\zeta^2 + \cos^2 \zeta d\psi_1^2}{1 - \rho^4 \sin^2 \zeta} + \sin^2 \zeta d\psi_2^2 \right], \quad (2.14)$$

$$ds_{S_5, \varkappa=i}^2 = d\varphi^2 + ds_{S\perp}^2, \quad ds_{S\perp}^2 = \frac{dr^2}{(1 - r^2)^2} + r^2 \left[\frac{d\theta^2 + \cos^2 \theta d\phi_1^2}{1 - r^4 \sin^2 \theta} + \sin^2 \theta d\phi_2^2 \right]. \quad (2.15)$$

¹³The mirror theory is found via a double Wick rotation on the world sheet in the light-cone gauge [39]. The mirror TBA was discussed for complex q equal to root of unity in [40].

Thus the 10d metric factorizes as $R_t \times S_\varphi^1 \times M_A^4 \times M_S^4$.¹⁴ The B -field in (2.6) becomes imaginary, implying that the resulting string action will represent an integrable but non-unitary theory. This is not surprising since for $\varkappa = i$ the q -parameter in (1.1) is complex and thus the corresponding light-cone S-matrix becomes non-unitary (cf. [14, 22, 41, 42, 11]).

Another indication that this limit is special is that the resulting light-cone S-matrix is *relativistically invariant*: the decoupling of the two directions t, φ implies that the light-cone gauge fixing is straightforward (as in flat and pp-wave space examples) and does not break 2d Lorentz invariance. As we shall see, this limit is closely related to the Pohlmeyer reduced model [18, 20] for the undeformed $AdS_5 \times S^5$ superstring which has a relativistic massive S-matrix.

As in the $\varkappa \rightarrow \infty$ case discussed above, there is, however, a subtlety to be addressed: in the formal substitution of $\varkappa = i$ into the metric we ignored the fact that the effective tension in (2.1) vanishes. To get a non-zero string action we thus need to either (i) rescale g (taking it to infinity as $\varkappa \rightarrow i$ so that \hat{T} in (2.1) stays finite) or (ii) keep g fixed and compensate \hat{T} going to zero by rescaling string coordinates (as in the standard pp-wave limit, see, e.g., [43]). Let us follow the first route but also correlate $\varkappa \rightarrow i$ with a rescaling of just t and φ in (2.2), (2.4). This allows us to define the $\varkappa \rightarrow i$ limit in a non-trivial way, so that the t, φ directions do not automatically decouple:

$$\varkappa^2 = -1 + s\epsilon^2, \quad t = \epsilon^{-1}x^+ - \epsilon x^-, \quad \varphi = \epsilon^{-1}x^+ + \epsilon x^-, \quad \epsilon \rightarrow 0, \quad (2.16)$$

where s is an arbitrary constant. Then from (2.2), (2.4), (2.6) we get the following pp-wave type 10d metric and B -field

$$ds^2 = 4dx^+dx^- - s[V(\alpha) + \tilde{V}(\beta)]dx^{+2} + ds_{A\perp}^2 + ds_{S\perp}^2, \quad (2.17)$$

$$V(\alpha) = \sin^2 \alpha, \quad \tilde{V}(\beta) = \sinh^2 \beta, \quad \rho \equiv \tan \alpha, \quad r \equiv \tanh \beta, \quad (2.18)$$

$$ds_{A\perp}^2 = d\alpha^2 + \tan^2 \alpha \left[\frac{d\zeta^2 + \cos^2 \zeta d\psi_1^2}{1 - \tan^4 \alpha \sin^2 \zeta} + \sin^2 \zeta d\psi_2^2 \right], \quad (2.19)$$

$$ds_{S\perp}^2 = d\beta^2 + \tanh^2 \beta \left[\frac{d\theta^2 + \cos^2 \theta d\phi_1^2}{1 - \tanh^4 \beta \sin^2 \theta} + \sin^2 \theta d\phi_2^2 \right], \quad (2.20)$$

$$B_{\psi_1\zeta} = i \frac{\tan^4 \alpha \sin \zeta \cos \zeta}{1 - \tan^4 \alpha \sin^2 \zeta}, \quad B_{\phi_1\theta} = -i \frac{\tanh^4 \beta \sin \theta \cos \theta}{1 - \tanh^4 \beta \sin^2 \theta}, \quad (2.21)$$

where the 4d “transverse” metrics (2.19), (2.20) are the same as in (2.14), (2.15) after the coordinate transformations. Fixing the light-cone gauge $x^+ = \mu\tau$ (and ignoring the fermions) then gives the direct sum of two bosonic relativistic interacting integrable massive models.¹⁵ Note that $s = 0$ is the case of the naive $\varkappa = i$ limit in (2.14), (2.15) where the light-cone gauge theory had no potential. The case of $s < 0$ leads to negative-definite potential, so in what follows we set $s = 1$.¹⁶

The resulting theory of 4 + 4 massive bosons looks very similar to the bosonic truncation of the generalized sine-Gordon model that appeared as the Pohlmeyer reduction (PR) of the

¹⁴The scalar curvature of M_A^4 is $R = \frac{2}{\rho^2} \left[2(1 - \rho^4 \sin^2 \zeta) - 4(\rho^2 + 1) - \frac{28(\rho^2 + 1)^2}{(1 - \rho^4 \sin^2 \zeta)^2} + \frac{6(3\rho^2 + 5)(\rho^2 + 1)}{1 - \rho^4 \sin^2 \zeta} \right]$.

¹⁵Expanding in small α and β gives a theory of 4+4 massive fields.

¹⁶The norm of s does not matter as it can be absorbed into a rescaling of x^+ and x^- or μ .

$AdS_5 \times S^5$ superstring [18]. One may wonder if there is a gauge fixing of the gWZW model representing the PR of the $AdS_5 \times S^5$ string [18] that leads directly to this light-cone theory. This may seem unlikely for several reasons: (i) the PR action is real, while the $\varkappa = i$ limit leads to an imaginary WZ term; (ii) the metric of the G/H gWZW model with non-abelian H , found after solving for the H -gauge field, should have no isometries [18], while here we have four $U(1)$ isometries; (iii) there is no B -field coupling in the gWZW model with maximal subgroup H gauged [18], while the $\varkappa = i$ limit leads to a non-zero (and imaginary) WZ term; (iv) the metric of the G/H gWZW model with non-abelian H does not admit a perturbation theory around a simple vacuum, while here there is a well-defined expansion around the $\alpha = \beta = 0$ point.

There are, however, hints of a close connection between the two theories. As observed earlier, to retain a finite string tension while taking $\varkappa \rightarrow i$ we also need to take $g \rightarrow \infty$. Therefore, it is natural to expect that the S-matrix for the massive excitations of the light-cone theory originating from (2.17) should be related to the strong coupling limit of the interpolating S-matrix [14, 16] with q as a phase.¹⁷ This S-matrix is not unitary, which is a reflection of the non-reality of the light-cone gauge Lagrangian mentioned above.

The connection between the interpolating S-matrix in this limit and the Pohlmeyer reduction of the $AdS_5 \times S^5$ superstring was observed in [15] and discussed in detail in [22, 41]. While there is no precise agreement, at tree-level the S-matrix of the PR theory is given by the parity-even (unitary) piece of the interpolating S-matrix.¹⁸ This can also be seen explicitly in the expansion to quartic order of the light-cone gauge theory corresponding to (2.17)–(2.21):

$$\begin{aligned}
L = & -\partial_i \alpha \partial^i \alpha - (\alpha^2 + \frac{2\alpha^4}{3}) \left(\partial_i \zeta \partial^i \zeta + \cos^2 \zeta \partial_i \psi_1 \partial^i \psi_1 + \sin^2 \zeta \partial_i \psi_2 \partial^i \psi_2 \right) - \mu^2 (\alpha^2 - \frac{\alpha^4}{3}) \\
& -\partial_i \beta \partial^i \beta - (\beta^2 - \frac{2\beta^4}{3}) \left(\partial_i \theta \partial^i \theta + \cos^2 \theta \partial_i \phi_1 \partial^i \phi_1 + \sin^2 \theta \partial_i \phi_2 \partial^i \phi_2 \right) - \mu^2 (\beta^2 + \frac{\beta^4}{3}) \\
& + i\alpha^4 \sin 2\zeta \epsilon^{ij} \partial_i \psi_1 \partial_j \zeta - i\beta^4 \sin 2\theta \epsilon^{ij} \partial_i \phi_1 \partial_j \theta + O(\alpha^6) + O(\beta^6) .
\end{aligned} \tag{2.22}$$

Introducing the fields

$$Z_1 + iZ_2 = \alpha \cos \zeta e^{i\psi_1} , \quad Z_3 + iZ_4 = \alpha \sin \zeta e^{i\psi_2} , \tag{2.23}$$

$$Y_1 + iY_2 = \beta \cos \theta e^{i\phi_1} , \quad Y_3 + iY_4 = \beta \sin \theta e^{i\phi_2} , \tag{2.24}$$

we find that the metric part of (2.22) (the first two lines) matches the quartic bosonic terms of the PR action in [21], and thus the corresponding tree-level S-matrix (with non-unitary B -field terms omitted) should match the tree-level PR theory S-matrix in [21]. At the same time, while the tree-level PR S-matrix of [21] did not satisfy the standard classical Yang-Baxter equation, the S-matrix corresponding to the above light-cone gauge theory with the imaginary B -field terms included will satisfy it (in agreement with the classical integrability [7] of the deformed theory for any \varkappa).

¹⁷There are different ways to take the strong coupling limit of the interpolating S-matrix [14, 16] (see, e.g. [15]). In particular, in [40] it was pointed out that, depending on the scaling of the world-sheet momentum, one can arrive at either a massless dispersion relation, which should correspond to $s = 0$ in (2.16), or the massive PR dispersion relation, corresponding to $s = 1$.

¹⁸Note that an alternative gauge fixing, bypassing the use of the metric of the G/H gWZW model, was used in [21] to set up a perturbative expansion around the trivial vacuum.

It is also interesting to note that deforming the AdS_5 metric and taking $\varkappa \rightarrow i$ gives rise to a model that is similar to the PR of the string on $R \times S^5$, and vice versa – the $\varkappa = i$ limit of the deformed S^5 is similar to the PR of the string on $AdS_5 \times S^1$. Thus the roles of AdS_5 and S^5 appear to be interchanged as we move from $\varkappa = 0$ to $\varkappa = i$. Interestingly, this is a feature that was observed in the S-matrix picture via an analysis of the scattering of bound states in the interpolating theory with q being a phase [44].

In [42] it was claimed that the S-matrix for the physical states of the PR model should be given by the vertex-to-IRF transformation of the interpolating S-matrix with q being a phase. The resulting S-matrix is unitary and also has the perturbative tree-level S-matrix of [21] as a limit. It remains to be seen if this transformation can be lifted precisely to a relation between the “pp-wave” model (2.17)–(2.21) and the PR model of [18].

As we shall demonstrate below in sections 3 and 4, upon dimensional reduction to deformations of the $AdS_3 \times S^3$ and $AdS_2 \times S^2$ theories the relation between the $\varkappa = i$ deformed string theory and the PR model for the undeformed theory becomes much more straightforward.

2.3 Consistent truncations to low-dimensional models

The bosonic part of the model (2.1)–(2.6) is classically integrable [7] and thus any consistent truncation of the corresponding string equations yields a classically integrable string model.

A 3d truncation of the metric in (2.2), (2.3) is found by setting $\zeta = \psi_2 = 0$, $\psi_1 \equiv \psi$:¹⁹

$$ds_{A_3}^2 = -h(\rho)dt^2 + f(\rho)d\rho^2 + \rho^2 d\psi^2 . \quad (2.25)$$

The scalar curvature of this metric is $R = -\frac{2[3+\varkappa^2-(3-\varkappa^2)\varkappa^2\rho^2+\varkappa^4\rho^4]}{1-\varkappa^2\rho^2}$. Similarly, from (2.4), (2.5) we get ($\phi_1 \equiv \phi$)

$$ds_{S_3}^2 = \tilde{h}(r)d\varphi^2 + \tilde{f}(r)dr^2 + r^2 d\phi^2 . \quad (2.26)$$

The B -field (2.6) vanishes, i.e. we get purely metric 3d integrable models that represent the \varkappa -deformations of AdS_3 and S^3 respectively.

Reducing further by setting $\psi = 0$ in (2.25) and $\phi = 0$ in (2.26) gives two 2d metrics

$$ds_{A_2}^2 = -\frac{1+\rho^2}{1-\varkappa^2\rho^2}dt^2 + \frac{d\rho^2}{(1+\rho^2)(1-\varkappa^2\rho^2)} , \quad (2.27)$$

$$ds_{S_2}^2 = \frac{1-r^2}{1+\varkappa^2r^2}d\varphi^2 + \frac{dr^2}{(1-r^2)(1+\varkappa^2r^2)} . \quad (2.28)$$

These are \varkappa -deformations of AdS_2 and S^2 which, like (2.25) and (2.26), are related by an obvious analytic continuation.

Let us note that since the bosonic model based on the deformed $S^5 = SO(6)/SO(5)$ metric (2.4) (or deformed AdS_5 metric (2.2)) can be obtained directly using the deformed coset construction of [8], the same applies to their truncations to lower-dimensional $SO(n+1)/SO(n)$

¹⁹Note that setting $\psi_1 = \psi_2 = 0$ is not a consistent truncation because of the B -field (2.6) contribution to the ψ_1 equation of motion.

cosets with $n = 3, 2$. Indeed, a metric equivalent to (2.28) was found in [8] as representing the deformed $SO(3)/SO(2)$ coset, and (2.26) should be the metric for the similarly deformed coset $SO(4)/SO(3)$.

We shall study the deformed $AdS_3 \times S^3$ and $AdS_2 \times S^2$ models in detail in sections 3 and 4. As we shall explain below, the models based on (2.28) and (2.26) are actually not new: they are well-known deformations of the S^2 and S^3 sigma models constructed in [27] and [25] respectively, and their classical integrability was proven earlier in [26].

An interesting open question is how to promote these classical integrable models to 10d conformal superstring sigma models that represent deformations of $AdS_3 \times S^3 \times T^4$ and $AdS_2 \times S^2 \times T^6$ models. This requires finding the dilaton and other fluxes that together with the above metrics solve the 10d type II supergravity equations of motion.

3 Deformed $AdS_3 \times S^3$ model

Let us now consider in detail the model whose bosonic part is given by the deformation of the $AdS_3 = SO(2, 2)/SO(1, 2)$ and $S^3 = SO(4)/SO(3)$ cosets, i.e. with a metric which is the sum of (2.25) and (2.26).

Since the two 3d metrics are related by an obvious analytic continuation let us concentrate on the structure of the \varkappa deformation of S^3 in (2.26), i.e.

$$ds_{S^3}^2 = \frac{dr^2}{(1-r^2)(1+\varkappa^2 r^2)} + r^2 d\phi^2 + \frac{1-r^2}{1+\varkappa^2 r^2} d\varphi^2. \quad (3.1)$$

Here ϕ and φ are two $U(1)$ isometry directions and, in addition to $U(1) \times U(1)$, this model also has a discrete Z_2 symmetry (the dr^2 term is invariant under this change)

$$\phi \leftrightarrow \varphi, \quad r \rightarrow \sqrt{\frac{1-r^2}{1+\varkappa^2 r^2}}. \quad (3.2)$$

A 3d metric with exactly the same symmetries is a special case of Fateev's [25] 2-parameter renormalizable deformation of the $SU(2)$ principal chiral model which is known to be classically integrable [26]: as we shall explain in subsection 3.1 below, the sigma model based on (3.1) is the same as the “symmetric” case of the Fateev model.

Furthermore, we shall demonstrate in Appendix A that the recently constructed 2-parameter family of integrable Yang-Baxter deformations of the principal chiral model for group G [10, 28] is equivalent to Fateev model in the $G = SU(2)$ case and thus also contains (3.1) as its special equal-parameter case.²⁰ Thus (3.1), which corresponds to the deformation of $SO(4)/SO(3)$ constructed according to [8], is at the same time a special case of the Fateev model [25] and also a special case of the bi-Yang-Baxter sigma model of [28].

The $AdS_3 \times S^3$ case is special compared to the $AdS_5 \times S^5$ and $AdS_2 \times S^2$ cosets as it also has an interpretation in terms of a product of group spaces. In this case the bosonic S^3 (and AdS_3) part has a *two*-parameter integrable deformation [25, 28], and there is also a further deformation [26]

²⁰That the coset deformation of [8] for the $SO(4)/SO(3)$ case is equivalent to the equal-parameter case of the $SU(2)$ model of [28] was mentioned to us by the authors of [7].

that includes a non-zero B -field coupling (i.e. a WZ term) which we shall discuss in Appendix B. This gives a 3-parameter deformation of the S^3 model with the extra parameter being the coefficient q of the WZ term (with $q = 0$ being the S^3 model and $q = 1$ corresponding to the $SU(2)$ WZW model). With the deformation parameters in the two bosonic factors identified, there should then exist the corresponding 2-parameter deformation of the $AdS_3 \times S^3$ supercoset model with mixed 3-form flux discussed in [30, 45]. This then suggests that there should exist an extension of the G, B background by the dilaton and RR fluxes that preserves conformal invariance. It should be noted, however, that already in the presence of the single-parameter κ -deformation the dilaton becomes non-trivial and thus S-duality transformations of the type IIB theory will change the sigma model (string-frame) metric. Consequently, there will no longer be any symmetry between the NS-NS and R-R choices of 2-form background.²¹ For that reason it would be best to study the 2-parameter deformations of the cases $q = 0$ and $q = 1$ separately. Apart from in Appendix B, here we will consider only the $q = 0$ case, i.e. without B -field coupling.

Like the $AdS_5 \times S^5$ model, the κ -deformed $AdS_3 \times S^3$ model admits two special cases: $\kappa = \infty$ and $\kappa = i$. Taking the limit $\kappa \rightarrow \infty$ in (3.1) and introducing $\bar{r} = r^{-1}$ and $\bar{\phi} = \kappa\phi$ we get $ds_{S^3}^2 = \kappa^{-2}[(\bar{r}^2 - 1)^{-1}d\bar{r}^2 + \bar{r}^{-2}d\bar{\phi}^2 + (\bar{r}^2 - 1)d\varphi^2]$. This becomes equivalent to the metric of euclidean AdS_3 space, i.e. the hyperboloid H^3 or euclidean $SL(2, R)$ group space, after T-duality in the $\bar{\phi}$ direction. Thus, in this sense, the κ -deformation relates the $AdS_3 \times S^3$ coset to $dS_3 \times H^3$. Interestingly, $\kappa = 0$ and $\kappa = \infty$ correspond to the two (IR and UV) asymptotics of the RG flow in the deformed S^3 model [26].²²

The $\kappa = i$ limit of the deformed $AdS_3 \times S^3$ coset will be discussed in subsection 3.2. We shall see that the metric takes the pp-wave form which in the light-cone gauge reduces to the sum of the complex sine-Gordon model and its analytic continuation, which is precisely the bosonic part of the Pohlmeyer reduced theory for $AdS_3 \times S^3$ (times T^4) superstring theory [18, 20]. Furthermore, we shall find the dilaton and RR 3-form flux which promote, as in [31, 32, 33], this metric to a type IIB supergravity solution. The fermionic part of the corresponding superstring Lagrangian is then found to match the fermionic part of the PR theory for the $AdS_3 \times S^3$ superstring [20]. Thus the $\kappa = i$ limit of the deformed $AdS_3 \times S^3$ supercoset should represent an exact embedding of the PR model of the undeformed ($\kappa = 0$) supercoset into string theory.

3.1 κ -deformed S^3 as the “symmetric” case of the Fateev model

Ref. [25] proposed a two-parameter deformation the $O(4)$ sigma model that (i) preserves $U(1) \times U(1)$ symmetry and (ii) is perturbatively renormalizable, i.e. that the change of the sigma model metric under the RG flow (a shift by its Ricci tensor at one-loop order) can be represented by a change of the two deformation parameters (and the overall S^3 “radius” coupling constant). The Lagrangian for the field $g \in SU(2)$ is [25]

$$L_{S^3} = \frac{1}{2[(1 + \ell)(1 + r) - \ell r M^2]} \eta^{ij} \left(\frac{1}{2} \text{Tr}[\partial_i g \partial_j g^{-1}] + \ell L_i^3 L_j^3 + r R_i^3 R_j^3 \right), \quad (3.3)$$

²¹Also, S-duality need not, in general, preserve the integrability of string sigma model, cf. [2, 3].

²²These are zeroes of the beta function for κ , but to have a fixed point for all couplings one needs a WZ term [26] (see also Appendix B).

where $i, j = 1, 2$ and (σ^a) are Pauli matrices)

$$M = \frac{1}{2} \text{Tr}[g\sigma^3 g^{-1}\sigma^3] , \quad L_i^a = \frac{1}{2i} \text{Tr}[\partial_i g g^{-1}\sigma^a] , \quad R_i^a = \frac{1}{2i} \text{Tr}[g^{-1}\partial_i g \sigma^a] . \quad (3.4)$$

In (3.3) ℓ and r are two independent deformation parameters, and we shall also use²³

$$d \equiv \frac{1}{2}(\ell + r) , \quad c \equiv \frac{1}{2}(\ell - r) . \quad (3.5)$$

One may parametrize the $SU(2)$ field as

$$g = n_4 \mathbb{1} + i n_a \sigma^a , \quad n_a n_a + n_4^2 = 1 , \quad (3.6)$$

$$n_1 + i n_2 = w e^{i\chi_1} , \quad n_3 + i n_4 = \sqrt{1 - w^2} e^{i\chi_2} , \quad (3.7)$$

$$z \equiv n_1^2 + n_2^2 - n_3^2 - n_4^2 , \quad w^2 = \frac{1}{2}(1 + z) . \quad (3.8)$$

Then the 3d target space metric for the two-parameter model (3.3) becomes [25]

$$ds_3^2 = U(z)dz^2 + D(z)d\chi_1^2 + D(-z)d\chi_2^2 + 2C(z)d\chi_1 d\chi_2 , \quad (3.9)$$

$$U(z) = (1 - z^2)^{-1} Q(z) , \quad Q \equiv \frac{1}{4} [(1 + d)^2 - c^2 - (d^2 - c^2)z^2]^{-1} , \quad (3.10)$$

$$D(z) = 2(1 + z)[1 + d(1 + z)]Q(z) , \quad C(z) = 2c(1 - z^2)Q(z) . \quad (3.11)$$

As we shall show in Appendix A, this model is equivalent to the $SU(2)$ case of the two-parameter integrable deformation of the principal chiral model constructed in [10, 28]. On the one hand, this gives a simpler demonstration of the integrability of the sigma model (3.9) than that in [26], and on the other, it proves the renormalizability of the 2-parameter model of [10, 28] (checked in the 1-parameter, $r = 0$, case in [46]).

There are two obvious special 1-parameter cases: left-right asymmetric ($\ell = 0$ or $r = 0$) and left-right symmetric ($\ell = r$). For $r = 0$ we get $d = c = \frac{1}{2}\ell$ and the metric (3.9) becomes that of the squashed S^3 corresponding to the anisotropic $SU(2)$ chiral model [29].

In the case of left-right symmetric deformation²⁴

$$\ell = r = d , \quad c = 0 , \quad (3.12)$$

the metric (3.9) simplifies to

$$ds^2 = U(z)dz^2 + D(z)d\chi_1^2 + D(-z)d\chi_2^2 , \quad (3.13)$$

$$U(z) = \frac{1}{4(1 - z^2)[(1 + d)^2 - d^2 z^2]} , \quad D(z) = \frac{1 + z}{2[(1 + d) - dz]} . \quad (3.14)$$

This metric is manifestly invariant under the Z_2 symmetry: $z \rightarrow -z$ and $\chi_1 \leftrightarrow \chi_2$. It is indeed equivalent to (3.1) under the following identification of the coordinates and parameters:

$$r^2 = \frac{1 + z}{2[(1 + d) - dz]} , \quad \phi = \chi_1 , \quad \varphi = \chi_2 , \quad \varkappa^2 = 4d(1 + d) . \quad (3.15)$$

²³We have extracted the overall scale so that the other the parameters used in [25] are $u = 1$, $a^2 = (1 + \ell)(1 + r)$, $b^2 = \ell r$.

²⁴Another special 1-parameter case is $\ell = -r = c$, $d = 0$ so that $Q(z) = \frac{1}{4}(1 - c^2 + c^2 z^2)^{-1}$, $U(z) = (1 - z^2)^{-1}Q$, $D(z) = 2(1 + z)^{-1}Q$, $C(z) = 2c(1 - z^2)Q$. Performing T-dualities along χ_1 and χ_2 we get the metric $ds^2 = \frac{dz^2}{4(1 - z^2)(1 - c^2 + c^2 z^2)} + \frac{2}{1 + z} d\tilde{\chi}_1^2 + \frac{2}{1 - z} d\tilde{\chi}_2^2 - 4cd\tilde{\chi}_1 d\tilde{\chi}_2$ and zero B -field coupling.

Note that $r^2 = \frac{1}{2}(1+z) + O(\varkappa^2)$, i.e. in the undeformed case r is same as w in (3.7) (cf. (3.8)).

Let us note that the metric (3.9) becomes flat for $d = -\frac{1}{2}$, i.e. $\ell + r = -1$. In this case we see from (3.15) that $\varkappa = i$, but the change of variables $r \rightarrow z$ degenerates, so there is no contradiction with the fact that the metric (3.1) remains non-trivial for $\varkappa = i$: rescaling z in (3.9) together with taking $d \rightarrow -\frac{1}{2}$ gives the metric in (3.1).

As was mentioned above, there should exist an integrable 2-parameter deformation of the $AdS_3 \times S^3$ supercoset model whose bosonic part is given by the direct product of the 2-parameter deformed S^3 (3.9) and the similarly deformed (with the same parameters) AdS_3 . To be sure that this deformation will represent a conformal model one needs to find the corresponding dilaton and type IIB fluxes promoting such a 3+3 dimensional metric to an exact type IIB supergravity solution. Finding such a solution at the moment is an open problem even in the case of the symmetric 1-parameter deformation.

3.2 $\varkappa = i$ limit: equivalence to the PR model for the $AdS_3 \times S^3$ superstring

To shed light on the underlying type IIB background and to establish a relation to the PR model for the $AdS_3 \times S^3$ superstring, let us now consider the special $\varkappa = i$ limit of the deformed $AdS_3 \times S^3$ model (2.25), (2.26). Directly setting $\varkappa = i$ in this deformed $AdS_3 \times S^3$ metric leads to the decoupling of the coordinates t and φ . Coupling this limit with a nontrivial rescaling of these coordinates as described in eq. (2.16) leads to a non-trivial pp-wave metric which is a truncation of (2.17) (we set $s = 1$)

$$ds^2 = 4dx^+dx^- - [V(\alpha) + V(\beta)]dx^{+2} + d\alpha^2 + \tan^2 \alpha d\psi^2 + d\beta^2 + \tanh^2 \beta d\phi^2, \quad (3.16)$$

$$V(\alpha) = \sin^2 \alpha, \quad V(\beta) = \sinh^2 \beta, \quad \rho \equiv \tan \alpha, \quad r \equiv \tanh \beta.$$

Fixing light-cone gauge, $x^+ = \mu\tau$, we find the following Lagrangian

$$L_{l.c.} = -(\partial_i \alpha \partial^i \alpha + \tan^2 \alpha \partial_i \psi \partial^i \psi + \mu^2 \sin^2 \alpha) - (\partial_i \beta \partial^i \beta + \tanh^2 \beta \partial_i \phi \partial^i \phi + \mu^2 \sinh^2 \beta), \quad (3.17)$$

which is precisely the bosonic part of the PR Lagrangian for strings moving on $AdS_3 \times S^3$ [18, 20]. Note that as in the $AdS_5 \times S^5$ case, the roles of the AdS_n and S^n spaces appear to be interchanged, i.e. the $\varkappa \rightarrow i$ limit of the deformed AdS_3 metric gives the PR of the string on $R \times S^3$ (after fixing light-cone gauge), and vice versa – the $\varkappa \rightarrow i$ limit of the deformed S^3 leads to the PR of the string on $AdS_3 \times S^1$.

Let us now extend the direct product of the pp-wave space (3.16) and a torus T^4 to a full solution of type IIB supergravity by finding the corresponding dilaton and 3-form RR flux that solve the equations (we shall assume that all other fluxes vanish)²⁵

$$R + 4\nabla^\mu \nabla_\mu \Phi - 4\nabla^\mu \Phi \nabla_\mu \Phi = 0, \quad (3.18)$$

$$tR_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi = t\frac{1}{4}e^{2\Phi}(F_\mu{}^{\rho\sigma}F_{\nu\rho\sigma} - \frac{1}{6}g_{\mu\nu}F^{\eta\rho\sigma}F_{\eta\rho\sigma}), \quad (3.19)$$

$$\partial_{[\mu}F_{\nu\eta\rho]} = 0, \quad \partial_\mu(\sqrt{-g}F^{\mu\nu\rho}) = 0. \quad (3.20)$$

²⁵Note that since F_3 will be assumed to be non-zero only in 6d subspace, its stress tensor is traceless (or, equivalently, $F^{\eta\rho\sigma}F_{\eta\rho\sigma} = 0$) and thus $R + 2\nabla^2 \Phi = 0$. Then the dilaton equation may be written as $\nabla^2 e^{-2\Phi} = 0$.

Let us change the coordinates to complex u, w so that (3.16) becomes

$$ds^2 = 4dx^+ dx^- - (|u|^2 + |w|^2) dx^{+2} + \frac{du d\bar{u}}{1 - |u|^2} + \frac{dw d\bar{w}}{1 + |w|^2} , \quad (3.21)$$

$$u = \sin \alpha e^{i\psi} , \quad w = \sinh \beta e^{i\phi} . \quad (3.22)$$

The Ricci scalar of this metric is non-zero because of the curved transverse space

$$R = \frac{4}{1 + |w|^2} - \frac{4}{1 - |u|^2} . \quad (3.23)$$

The solution to the dilaton equation can be written as

$$\Phi = -\ln f(|u|^2) - \ln g(|w|^2) , \quad (3.24)$$

$$\begin{aligned} f(x) &= \sqrt{1-x} [c_1 P_v(-1+2x) + c_2 Q_v(-1+2x)] , & v = \frac{1}{2}(c_0 - 1) , \\ g(x) &= \sqrt{1+x} [c_3 P_v(1+2x) + c_4 Q_v(1+2x)] . \end{aligned} \quad (3.25)$$

Here c_n are integration constants, c_0 is the separation constant appearing when splitting equation (3.18) as $E(u) = c_0$, $E(w) = -c_0$. P_v and Q_v are the two independent solutions of the Legendre equation, where Q_v has logarithmic singularities and can thus be ignored. P_v is a polynomial if the index v is an integer. The simplest choice is $v = 0$ ($c_0 = 1$) when $P_0 = \text{const}$ and thus

$$\text{t}\Phi = \Phi_0 - \frac{1}{2} \ln(1 - |u|^2) - \frac{1}{2} \ln(1 + |w|^2) . \quad (3.26)$$

The transverse metric in (3.21) and the dilaton (3.26) represent of course the direct sum of the familiar 2d black hole ($SL(2, R)/U(1)$) solution and its analytic continuation. It follows from (3.26) that $R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi$ has only one non-zero component

$$R_{++} + 2\nabla_+ \nabla_+ \Phi = 4(1 - |u|^2 + |w|^2) . \quad (3.27)$$

It is readily seen that it can be balanced by the solution for the RR 3-form flux $F_{\mu\nu\rho}$ with the following real potential²⁶

$$C_2 = it \frac{1}{\sqrt{2}} e^{-\Phi_0} [\cos \gamma (1 + |w|^2)(ud\bar{u} - \bar{u}du) + \sin \gamma (1 - |u|^2)(wd\bar{w} - \bar{w}dw)] \wedge dx^+ , \quad (3.28)$$

where γ is a free parameter. Motivated by matching the Pohlmeyer reduction of the $AdS_3 \times S^3$ string [20], which has a formal Z_2 symmetry interchanging the AdS_3 and S^3 parts, a natural choice for γ is $\frac{\pi}{4}$, i.e.

$$\cos \gamma = \sin \gamma = t \frac{1}{\sqrt{2}} . \quad (3.29)$$

²⁶We use the following conventions: $F_{3\mu\nu\rho} = (dC_2)_{\mu\nu\rho} = \partial_\mu C_{2\nu\rho} + \partial_\nu C_{2\rho\mu} + \partial_\rho C_{2\mu\nu}$, $C_2 = \frac{1}{2} C_{2\mu\nu} dx^\mu \wedge dx^\nu$, $F_3 = \frac{1}{6} F_{3\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho$.

The above pp-wave type background $M^6 \times T^4$ thus represents the embedding of the direct sum of the complex sine-Gordon model and its analytic continuation (3.17) into 10d type IIB string theory and thus belongs to the class of models discussed in [31, 32, 33].²⁷

It is straightforward to find the quadratic fermionic term in the GS superstring action corresponding to the above pp-wave background

$$L_{F2} = i(\eta^{ij}\delta^{IJ} + \epsilon^{ij}\sigma_3^{IJ})\bar{\theta}^I \not{\partial}_i D_j^{JK} \theta^K, \quad \not{\partial}_i = \Gamma_A e_M^A(X) \partial_i X^M, \quad (3.30)$$

$$D_i^{JK} = \partial_i X^M D_M^{JK}, \quad D_M^{JK} = (\partial_M + \frac{1}{4} \omega_M^{AB} \Gamma_{AB}) \delta^{JK} + \frac{1}{8} e^\phi \not{F}_{(3)} \Gamma_M \sigma_1^{JK}. \quad (3.31)$$

Here $\not{F}_{(n)} = \frac{1}{n!} F_{A_1 \dots A_n} \Gamma^{A_1 \dots A_n}$ and we assume conformal gauge. The fermions θ^I ($I = 1, 2$) are the two IIB Majorana-Weyl spinors, and σ_1^{IJ} , σ_3^{IJ} are Pauli matrices.²⁸ The explicit form of the product of the F_3 form corresponding to (3.28) and the dilaton factor that enters the covariant derivative is

$$e^\phi F_3 = i \left[\sqrt{1+|w|^2} \sqrt{1-|u|^2} (e^2 \wedge e^{\bar{2}} + e^4 \wedge e^{\bar{4}}) + \bar{u}\bar{w} e^2 \wedge e^4 - uw e^{\bar{2}} \wedge e^{\bar{4}} \right] \wedge e^+. \quad (3.32)$$

Fixing the light-cone gauge

$$x^+ = \mu\tau, \quad \Gamma^+ \theta^I = 0, \quad (3.33)$$

and rescaling the fermions by $\frac{1}{\sqrt{\mu}}$ we find $(\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1))$

$$\begin{aligned} L_{F2} = & i\bar{\theta}^1 \Gamma^- \left[\partial_- + \frac{1}{8} \frac{u\partial_- \bar{u} - \bar{u}\partial_- u}{1-|u|^2} \Gamma^{2\bar{2}} - \frac{1}{8} \frac{w\partial_- \bar{w} - \bar{w}\partial_- w}{1+|w|^2} \Gamma^{4\bar{4}} \right] \theta^1 \\ & + i\bar{\theta}^2 \Gamma^- \left[\partial_+ + \frac{1}{8} \frac{u\partial_+ \bar{u} - \bar{u}\partial_+ u}{1-|u|^2} \Gamma^{2\bar{2}} - \frac{1}{8} \frac{w\partial_+ \bar{w} - \bar{w}\partial_+ w}{1+|w|^2} \Gamma^{4\bar{4}} \right] \theta^2 \\ & + t \frac{1}{8} \mu \bar{\theta}^I \Gamma^- \left[\sqrt{1+|w|^2} \sqrt{1-|u|^2} (\Gamma^{2\bar{2}} + \Gamma^{4\bar{4}}) + \bar{u}\bar{w} \Gamma^{24} - uw \Gamma^{\bar{2}\bar{4}} \right] \sigma_1^{IK} \theta^K, \end{aligned} \quad (3.34)$$

where we have used that $\{\Gamma^+, \Gamma^-\} = -2\eta^{+-} = 4$. Returning to the original coordinates in (3.17) (cf. (3.22)) this can be rewritten as²⁹

$$\begin{aligned} L_{F2} = & i\bar{\theta}^1 \Gamma^- \left[\partial_- - \frac{1}{2} \tan^2 \alpha \partial_- \psi_1 \Gamma^{23} + \frac{1}{2} \tanh^2 \beta \partial_- \phi_1 \Gamma^{45} \right] \theta^1 \\ & + i\bar{\theta}^2 \Gamma^- \left[\partial_+ - \frac{1}{2} \tan^2 \alpha \partial_+ \psi_1 \Gamma^{23} + \frac{1}{2} \tanh^2 \beta \partial_+ \phi_1 \Gamma^{45} \right] \theta^2 \\ & - t \frac{i}{4} \mu \bar{\theta}^I \Gamma^- \left[\cos \alpha \cosh \beta (\Gamma^{23} + \Gamma^{45}) \right. \\ & \quad \left. - \sin \alpha \sinh \beta [\cos(\psi_1 + \phi_1) (\Gamma^{25} + \Gamma^{34}) - \sin(\psi_1 + \phi_1) (\Gamma^{24} - \Gamma^{35})] \right] \sigma_1^{IK} \theta^K. \end{aligned} \quad (3.35)$$

²⁷The embedding of this integrable model considered in [33] used a particular F_5 background instead of the F_3 background considered here.

²⁸We use the metric $\eta_{+-} = -\frac{1}{2}$, $\eta_{2\bar{2}} = \eta_{4\bar{4}} = \frac{1}{2}$, $\eta_{ij} = \delta_{ij}$ and $e^+ = dx^+$, $e^- = 4dx^- + (|u|^2 + |w|^2)dx^+$, $e^i = dz^i$, $e^2 = \frac{du}{\sqrt{1-|u|^2}}$, $e^{\bar{2}} = \frac{d\bar{u}}{\sqrt{1-|u|^2}}$, $e^4 = \frac{dw}{\sqrt{1+|w|^2}}$, $e^{\bar{4}} = \frac{d\bar{w}}{\sqrt{1+|w|^2}}$. The non-zero components of the spin connection are $\omega_{2\bar{2}} = -\omega_{2\bar{2}} = \frac{1}{4} \frac{u d\bar{u} - \bar{u} du}{1-|u|^2}$, $\omega_{4\bar{4}} = -\omega_{4\bar{4}} = -\frac{1}{4} \frac{w d\bar{w} - \bar{w} dw}{1+|w|^2}$.

²⁹We use that $\Gamma^{2\bar{2}} = \Gamma^{2+i3, 2-i3} = -2i\Gamma^{23}$, $\Gamma^{4\bar{4}} = \Gamma^{4+i5, 4-i5} = -2i\Gamma^{45}$, $\Gamma^{24} = \Gamma^{2+i3, 4+i5} = \Gamma^{24} - \Gamma^{35} + i\Gamma^{25} + i\Gamma^{34}$, etc.

Since the bosonic part of the light-cone gauge action is precisely the bosonic part of the action of the Pohlmeyer reduction of the $AdS_3 \times S^3$ superstring [20], it is natural to expect that the fermionic parts also match. To compare (3.35) with quadratic fermionic term in the PR action in [20] let us first decompose the fermions as

$$\theta^I = \theta_{||}^I + \theta_{\perp}^I, \quad (1 - \Gamma_{2345})\theta_{\perp}^I = 0, \quad (1 + \Gamma_{2345})\theta_{||}^I = 0. \quad (3.36)$$

$\theta_{||}^I$ and θ_{\perp}^I are decoupled at the quadratic level and we find

$$\begin{aligned} L_{F2}(\theta_{\perp}, \theta_{||}) &= L_{F2}(\theta_{\perp}) + i\bar{\theta}_{||}^1 \Gamma^{-} \left[\partial_{-} - \frac{1}{2}(\tan^2 \alpha \partial_{-} \psi_1 - \tanh^2 \beta \partial_{-} \phi_1) \Gamma^{23} \right] \theta_{||}^1 \\ &\quad + i\bar{\theta}_{||}^2 \Gamma^{-} \left[\partial_{+} - \frac{1}{2}(\tan^2 \alpha \partial_{+} \psi_1 - \tanh^2 \beta \partial_{+} \phi_1) \Gamma^{23} \right] \theta_{||}^2 \\ &\quad - t \frac{i}{2} \mu \bar{\theta}_{||}^I \Gamma^{-} \left[\cos \alpha \cosh \beta \Gamma^{23} - \sin \alpha \sinh \beta \left[\cos(\psi_1 + \phi_1) \Gamma^{25} - \sin(\psi_1 + \phi_1) \Gamma^{24} \right] \right] \sigma_1^{IK} \theta_{||}^K. \end{aligned} \quad (3.37)$$

To make contact with [20] we should choose a particular representation for the 10d Dirac matrices and also a solution to the light-cone gauge condition (3.33) and the orthogonal decomposition (3.36) (different choices may lead to different identifications of the components of θ_m^I , $m = 1, \dots, 32$, and the fermions in [20]). In the Majorana representation of the 10d Dirac matrices and with left-handed fermions one finds that only θ_m^I with $m = 29, 30, 31, 32$ survive the various projections and the action becomes:

$$\begin{aligned} L_{F2} &= L_{F2}(\theta_{\perp}) + 8i \left\{ \theta_{29}^1 \partial_{+} \theta_{29}^1 + \theta_{30}^1 \partial_{+} \theta_{30}^1 + \theta_{31}^1 \partial_{+} \theta_{31}^1 + \theta_{32}^1 \partial_{+} \theta_{32}^1 \right. \\ &\quad + \theta_{29}^2 \partial_{-} \theta_{29}^2 + \theta_{30}^2 \partial_{-} \theta_{30}^2 + \theta_{31}^2 \partial_{-} \theta_{31}^2 + \theta_{32}^2 \partial_{-} \theta_{32}^2 \\ &\quad - \tan^2 \alpha \left[\partial_{-} \psi_1 (\theta_{29}^1 \theta_{30}^1 + \theta_{31}^1 \theta_{32}^1) + \partial_{+} \psi_1 (\theta_{29}^2 \theta_{30}^2 + \theta_{31}^2 \theta_{32}^2) \right] \\ &\quad + \tanh^2 \beta \left[\partial_{-} \phi_1 (\theta_{29}^1 \theta_{30}^1 + \theta_{31}^1 \theta_{32}^1) + \partial_{+} \phi_1 (\theta_{29}^2 \theta_{30}^2 + \theta_{31}^2 \theta_{32}^2) \right] \\ &\quad - \mu \left[\cos \alpha \cosh \beta (-\theta_{30}^2 \theta_{29}^1 + \theta_{29}^2 \theta_{30}^1 - \theta_{32}^2 \theta_{31}^1 + \theta_{31}^2 \theta_{32}^1) \right. \\ &\quad - \sin \alpha \sinh \beta \cos(\psi_1 + \phi_1) (-\theta_{32}^2 \theta_{29}^1 + \theta_{29}^2 \theta_{30}^1 - \theta_{31}^2 \theta_{30}^1 + \theta_{30}^2 \theta_{31}^1) \\ &\quad \left. - \sin \alpha \sinh \beta \sin(\psi_1 + \phi_1) (+\theta_{31}^2 \theta_{29}^1 - \theta_{29}^2 \theta_{31}^1 + \theta_{30}^2 \theta_{32}^1 - \theta_{32}^2 \theta_{30}^1) \right] \Big\}. \end{aligned} \quad (3.38)$$

Eq. (3.38) can then be mapped to the quadratic fermionic term in the Lagrangian of the PR model for the $AdS_3 \times S^3$ superstring [20] by identifying the fields as follows³⁰

$$\begin{aligned} \theta_{29}^1 &= \frac{1}{2} \alpha, & \theta_{30}^1 &= \frac{1}{2} \beta, & \theta_{31}^1 &= \frac{1}{2} \delta, & \theta_{32}^1 &= \frac{1}{2} \gamma, \\ \theta_{29}^2 &= -\frac{1}{2} \sigma, & \theta_{30}^2 &= \frac{1}{2} \rho, & \theta_{31}^2 &= \frac{1}{2} \lambda, & \theta_{32}^2 &= -\frac{1}{2} \nu. \end{aligned} \quad (3.39)$$

The full light-cone gauge GS action for the background (3.16), (3.26), (3.28) also contains quartic fermionic terms (whose presence is related to the curvature of the transverse space), and there is no doubt that they should also match the quartic fermionic terms in the PR action [20].

³⁰ Here α, β , etc., are the fermionic fields used in [20]. Note also that $\partial_{\pm}^{\text{here}} = \frac{1}{2}(\partial_0 \pm \partial_1) = \frac{1}{2}\partial_{\pm}^{\text{there}}$.

We conclude that the GS string with the $\varkappa = i$ metric as bosonic part, which should be the $\varkappa = i$ limit of the deformed $AdS_3 \times S^3$ supercoset action (see footnote 5), should represent the embedding of the “massive” Pohlmeyer reduced model for the undeformed ($\varkappa = 0$) $AdS_3 \times S^3$ supercoset into superstring theory.

Let us note that, unlike the pp-wave solutions with flat transverse space, the type IIB solution (3.16), (3.26), (3.28) does not have residual target-space supersymmetry (the same is true also for the F_5 -flux supported background in [33]). One may nevertheless expect the existence of hidden (4, 4) world-sheet supersymmetry in the corresponding superstring theory [33]. This parallels the discussions in [18, 20, 47, 22, 48, 49] where it was argued that the PR theory for the $AdS_3 \times S^3$ superstring should have hidden supersymmetry.

4 Deformed $AdS_2 \times S^2$ model

Let us now consider the deformation of the $AdS_2 \times S^2$ coset with the bosonic part given by the sigma model corresponding to the metrics (2.27) and (2.28), i.e.

$$L = -\frac{1 + \rho^2}{1 - \varkappa^2 \rho^2} (\partial_i t)^2 + \frac{1}{(1 - \varkappa^2 \rho^2)(1 + \rho^2)} (\partial_i \rho)^2 \\ + \frac{1 - r^2}{1 + \varkappa^2 r^2} (\partial_i \varphi)^2 + \frac{1}{(1 + \varkappa^2 r^2)(1 - r^2)} (\partial_i r)^2. \quad (4.1)$$

The 1-parameter deformed S^2 model appearing here was first considered in [27] (with its integrability shown in [26]) and then rederived as a special case of the 1-parameter coset deformation construction in [8].

In the special limit $\varkappa \rightarrow \infty$ the Lagrangian (4.1) directly reduces (up to overall \varkappa^{-2} factor) to that of the 2d de Sitter plus 2d hyperboloid ($dS_2 \times H^2$) model (without an additional T-duality required in $AdS_5 \times S^5$ case in section 2.1). In the $\varkappa = i$ limit (4.1) the target space metric becomes flat; combining this limit with an additional rescaling of coordinates as in (2.16) then leads to a 4d pp-wave model which is a truncation of (2.17) or (3.16).

Below we shall first show how (4.1) emerges as the bosonic part of the deformed $AdS_2 \times S^2$ supercoset action constructed using the same method as in [7] and then discuss the special case of $\varkappa = i$.

4.1 Deformed supercoset Lagrangian

The superstring theory in $AdS_2 \times S^2 \times T^6$ is closely related [50] to the GS model based on the supercoset [24]

$$\frac{PSU(1, 1|2)}{SO(1, 1) \times U(1)} \quad (4.2)$$

which belongs to the class of supercosets in which the denominator is the fixed point of a Z_4 automorphism of the numerator. It moreover turns out that the construction of the fermionic

part of the action is quite sensitive to the Z_4 action. A possible choice for its generator that proves to be particularly useful, given in Appendix C of [18], is³¹

$$\omega(M) = \begin{pmatrix} -\sigma_3 M^{(a)t} \sigma_3 & \sigma_3 M^{(f_2)t} \sigma_3 \\ -\sigma_3 M^{(f_1)t} \sigma_3 & -\sigma_3 M^{(s)t} \sigma_3 \end{pmatrix}, \quad M = \begin{pmatrix} M^{(a)} & M^{(f_1)} \\ M^{(f_2)} & M^{(s)} \end{pmatrix}. \quad (4.3)$$

This ω identifies the gauge group generators as $\text{diag}(\sigma_1, 0)$ and $\text{diag}(0, i\sigma_1)$.

The GS Lagrangian for this supercoset may be written as³²

$$L_0 = \pi^{ij} \text{STr}[J_i d_0 J_j], \quad \pi^{ij} \equiv \sqrt{-g} g^{ij} - \epsilon^{ij}, \quad (4.4)$$

$$J_i = g^{-1} \partial_i g, \quad d_0 \equiv P_1 + 2P_2 - P_3, \quad (4.5)$$

where P_k are projectors onto subspaces with eigenvalue i^k under the action of the Z_4 automorphism.

The one-parameter η -deformation of this supercoset Lagrangian constructed according to [7] is^{33,34}

$$L = c_\eta \pi^{ij} \text{STr}[J_i d_\eta \circ \frac{1}{1 - \eta R_g \circ d_\eta} J_j], \quad (4.6)$$

$$d_\eta \equiv P_1 + 2c_\eta^{-1} P_2 - P_3, \quad c_\eta \equiv 1 - \eta^2. \quad (4.7)$$

The operator R_g acts on the superalgebra as

$$R_g(M) = g^{-1} R(g M g^{-1}) g, \quad (4.8)$$

where the operator R multiplies the generators corresponding to the positive roots by $-i$, those corresponding to the negative roots by $+i$ and annihilates the Cartan generators. It is possible to choose the positive roots to be generators whose nonzero entries are above the diagonal, which corresponds to considering the distinguished Dynkin diagram for $PSU(1, 1|2)$.³⁵

Independently of the choice of Z_4 automorphism, a systematic approach to expanding the Lagrangian in terms of coordinate fields is to represent the action of the operators d and R_g in the adjoint representation. We also introduce two auxiliary matrices \tilde{A} and \hat{A} :

$$d_\eta(T^a) = d_\eta^a{}_b T^b, \quad R(T^a) = R^a{}_b T^b, \quad g M g^{-1} = M_a \tilde{A}^a{}_b T^b, \quad g^{-1} M g = M_a \hat{A}^a{}_b T^b, \quad (4.9)$$

where M is a generic element of the algebra of $PSU(1, 1|2)$. Then

$$L = c_\eta \pi^{ij} J_{ia} J_{jd} \Omega^d{}_u d_\eta^u{}_v g^{av} \quad \Omega^{-1} \equiv \mathbb{1} - \eta d_\eta \tilde{A} R \hat{A}. \quad (4.10)$$

³¹ ω can be written in the form $\omega(M) = -K^{-1} M^{st} K$, where $K = \text{diag}(\sigma_3, \sigma_3)$ and st denotes the supertranspose: $\begin{pmatrix} M^{(a)} & M^{(f_1)} \\ M^{(f_2)} & M^{(s)} \end{pmatrix}^{st} = \begin{pmatrix} M^{(a)t} & -M^{(f_2)t} \\ M^{(f_1)t} & M^{(s)t} \end{pmatrix}$. Furthermore, this implies that ω satisfies $\omega(MN) = -\omega(N)\omega(M)$.

³²We use the normalization in which the (super)trace of squares of the bosonic Cartan generators equals 2.

³³Recall that in terms of \varkappa in (1.1) we have $\eta = \varkappa^{-1}[\sqrt{1 + \varkappa^2} - 1]$.

³⁴The action (2.1) corresponding to the Lagrangian in (4.6) is normalized as in [11].

³⁵The relation between the deformations corresponding to different choices of Dynkin diagram is unclear.

Using that $g^{ae} d_{\eta^c}^e = g^{ce} d_{\eta^a}^e$ the Lagrangian can be written in terms of $\Omega_{cg}^{be} \equiv \Omega^{be}$, where Ω^{be} may be interpreted as a deformation of the group-invariant metric $g^{be} = \text{STr}[T^b T^e]$, to which it reduces in the limit $\eta \rightarrow 0$.

We parametrize the coset elements as [11]

$$g = g_B g_F, \quad g_B = \begin{pmatrix} g_A & 0_{2 \times 2} \\ 0_{2 \times 2} & g_S \end{pmatrix}, \quad g_F = \exp \begin{pmatrix} 0_{2 \times 2} & f_1 \\ f_2 & 0_{2 \times 2} \end{pmatrix}, \quad (4.11)$$

$$g_A = e^{\frac{1}{2} i t \sigma_3} \begin{pmatrix} \rho_+ & i \rho_- \\ -i \rho_- & \rho_+ \end{pmatrix}, \quad \rho_{\pm} = \frac{1}{\sqrt{2}} \sqrt{\sqrt{\rho^2 + 1} \pm 1}, \quad (4.12)$$

$$g_S = e^{\frac{1}{2} i \varphi \sigma_3} \begin{pmatrix} r_+ & r_- \\ -r_- & r_+ \end{pmatrix}, \quad r_{\pm} = \frac{1}{\sqrt{2}} \sqrt{1 \pm \sqrt{1 - r^2}}, \quad (4.13)$$

with the fermions f_1 and f_2 related by a reality condition. The bosonic part of the Lagrangian (4.10), found by setting the fermions to zero, then coincides with (4.1), which is a truncation of the deformed $AdS_5 \times S^5$ model.

Using the parametrization of g_F given in eq. (C.4) of Appendix C we can explicitly construct the part of the deformed action quadratic in fermions. The resulting expression is rather lengthy, hence we will not present it here. However, it is useful to perform a simple test of one-loop UV finiteness of the deformed sigma model (4.4) by expanding to quadratic order in fields around the BMN type geodesic $t = \varphi = \tau$ of the deformed metric in (4.1). Doing so we find the following quadratic bosonic Lagrangian in conformal gauge³⁶

$$L = -4\partial_+ \tilde{t} \partial_- \tilde{t} + 4\partial_+ \tilde{\varphi} \partial_- \tilde{\varphi} + 4\partial_+ \rho \partial_- \rho - \frac{(1 + \eta^2)^2}{(1 - \eta^2)^2} \rho^2 + 4\partial_+ r \partial_- r - \frac{(1 + \eta^2)^2}{(1 - \eta^2)^2} r^2, \quad (4.14)$$

while the Lagrangian quadratic in fermions (again with a rescaling of the fields) is

$$L = -q_1 \partial_+ q_1 - s_1 \partial_- s_1 - \frac{1 + \eta^2}{1 - \eta^2} q_1 s_1 - q_2 \partial_+ q_2 - s_2 \partial_- s_2 - \frac{1 + \eta^2}{1 - \eta^2} q_2 s_2. \quad (4.15)$$

Thus, apart from the unphysical (longitudinal) fluctuations \tilde{t} and $\tilde{\varphi}$, we get 2 bosonic and 2 fermionic excitations with the same mass $\frac{1 + \eta^2}{1 - \eta^2} = \sqrt{1 + \varkappa^2}$. As a result, the one-loop partition function is finite.

4.2 Special case of $\varkappa = i$

Taking the limit $\varkappa \rightarrow i$ as in (2.16), (2.17) we get the following 4d pp-wave metric (cf. (3.16))³⁷

$$ds^2 = 4dx^+ dx^- - V(\alpha, \beta) dx^{+2} + d\alpha^2 + d\beta^2, \quad (4.16)$$

$$V = \sin^2 \alpha + \sin^2 \beta = |\zeta(v)|^2, \quad \zeta(v) = \sin v, \quad v = \alpha + i\beta. \quad (4.17)$$

³⁶Note we have rescaled the fields to put the Lagrangian in canonical form. Recall that here we use $\partial_{\pm} = \frac{1}{2}(\partial_0 \pm \partial_1)$.

³⁷Note that taking the limit $\varkappa = i$ directly in (4.1), i.e. without the rescaling of the coordinates as in (2.16), gives a flat space 4d model. In general, the scalar curvatures of the 2d metrics (2.27) and (2.28) in (4.1) are $R_A = -2(1 + \varkappa^2) \frac{1 + \varkappa^2 \rho^2}{1 - \varkappa^2 \rho^2}$ and $R_S = 2(1 + \varkappa^2) \frac{1 - \varkappa^2 r^2}{1 + \varkappa^2 r^2}$, i.e. they vanish for $\varkappa = i$. While the curvature is invariant under coordinate transformation in (2.16), this vanishing is still in agreement with the fact that the resulting 4d pp-wave metric has flat transverse part.

As in [31, 32, 33] this pp-wave metric can be promoted to a string solution with constant dilaton by adding a 4d vector field background (which may be viewed as an effective reduction of an RR field strength in the 10d space $M^4 \times T^6$)

$$A = [\zeta(v) + \bar{\zeta}(\bar{v})]dx^+ , \quad F = dA = [\zeta'(v)dv + \bar{\zeta}'(\bar{v})d\bar{v}]dx^+ . \quad (4.18)$$

This F solves Maxwell's equations and we also have $R_{++} = \frac{1}{2}\partial_r\partial_r V = F_{+v}F_{+\bar{v}}$. This background preserves 4d space-time supersymmetry.

The resulting light-cone gauge string action has a bosonic part which is the same as the bosonic part of the PR action for the $AdS_2 \times S^2$ superstring model [18, 20]. Furthermore, as in the $AdS_5 \times S^5$ and $AdS_3 \times S^3$ cases the roles of the AdS_n and S^n spaces are interchanged as we interpolate from $\varkappa = 0$ to $\varkappa = i$.

The full PR action is the same as that of the (2,2) world-sheet supersymmetric sine-Gordon model [18]. An equivalent action should be found from the GS action in the RR background (4.18) in the light-cone gauge (the term quartic in fermions vanishes in the light-cone gauge). The same fermionic terms come out of the light-cone gauge fixed supercoset Lagrangian (4.10) computed in the light-cone kappa-symmetry gauge. We include some details of the derivation in Appendix C where we also discuss the naive $\varkappa = i$ limit which leads to a flat space theory.³⁸

We conclude that the GS string theory with bosonic part given by the deformed $AdS_2 \times S^2$ at $\varkappa = i$ is equivalent to the deformed $AdS_2 \times S^2$ supercoset model with $\varkappa = i$ (see footnote 5) and represent an effective embedding of the massive integrable Pohlmeyer reduced model for the $AdS_2 \times S^2$ superstring into string theory.

5 Concluding remarks

In this paper we explored some limits and low-dimensional analogs of the deformed $AdS_5 \times S^5$ supercoset integrable model constructed in [7] with no explicit supersymmetry but with classical quantum group symmetry.

A remarkable feature of this model is the relation [11] of the corresponding light-cone gauge S-matrix to the real q deformed S-matrix [14, 16]. The latter also interpolates [22, 41, 16] (for q being a root of unity) between the non-relativistic $AdS_5 \times S^5$ “magnon” S-matrix and the massive relativistic S-matrix of the Pohlmeyer reduced model for the $AdS_5 \times S^5$ superstring.

We have studied the deformations of the low-dimensional $AdS_3 \times S^3$ and $AdS_2 \times S^2$ models and in these cases made the relation to the Pohlmeyer reduced theory explicit at the Lagrangian level. This was demonstrated by showing that in the $\varkappa = i$ limit the deformed model reduces to a certain pp-wave model that, in light-cone gauge, becomes equivalent to the generalized sine-Gordon model representing the PR theory for the undeformed supercoset. The details of a similar Lagrangian relation in the $AdS_5 \times S^5$ case and the issue of (non-)unitarity in the $\varkappa = i$ limit remain to be clarified.

We have also pointed out the possible existence of multiparameter deformations of the $AdS_3 \times S^3$ supercoset, clarifying the relation between the deformed S^3 bi-Yang-Baxter model of [10, 28] and the Fateev model [25] which itself is a special case of the integrable 3d model found in [26].

³⁸It appears that depending on the choice of Dynkin diagram, taking the limit $\varkappa \rightarrow i$ in (4.10) may require a rescaling of the fermionic variables.

Among many other open questions, it would be important to understand the meaning of the deformations suggested in [9, 10, 8, 7] at a path integral level. That may help confirm that the deformed $AdS_5 \times S^5$ model of [7] preserves the conformal invariance (as suggested by its classical kappa symmetry) and thus that the corresponding target space background solves the type IIB string Weyl invariance conditions. In fact, we have already provided several strong tests of the UV finiteness of the deformed supercoset model: (i) at $\varkappa = \infty$ it is related to the finite $dS_5 \times H^5$ model; (ii) at $\varkappa = i$ it is related to a finite pp-wave model representing the superstring embedding of the PR model; (iii) in the deformed $AdS_2 \times S^2$ case in section 4.1 we explicitly checked the one-loop UV finiteness by expanding near a BMN-type geodesic. It would nevertheless be useful to confirm the one-loop finiteness of the model [7] for generic \varkappa and generic world-sheet background.

Assuming the deformed $AdS_5 \times S^5$ model is one-loop finite, there is still a question about higher loop orders, i.e. the inverse string tension $\alpha' \sim T_0^{-1} = g^{-1}$ corrections. While hidden higher symmetries of the model of [7] may guarantee that its structure is preserved by divergent (local) loop corrections, to maintain it as a solution of the type IIB superstring Weyl invariance conditions one may need to deform the parameter \varkappa order by order in $1/g$ (starting with 4-loop $\alpha'^3 \sim g^{-3}$ order). If this happens, then the two parameters that enter the exact quantum light-cone S-matrix may be non-trivial functions of \varkappa and g appearing in the classical string action (2.1). In this case the semiclassical expression for q [11] in (1.1) may require a modification.

While this paper was in preparation we received [52] which discusses a similar construction to [7] except with the generalized sine-Gordon model as its starting point and then interpolating to the Hamiltonian for the light-cone gauge superstring. Clarifying the relation between [52] and [7] may help to understand this interpolation and the related issue of unitarity.

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A Equivalence of 2-parameter $SU(2)$ Yang-Baxter sigma model to Fateev model

It was shown in [10, 28] that the Lagrangian for $g \in G$

$$L_K = (\eta^{ij} + \epsilon^{ij}) \text{Tr} \left[J_i \frac{1}{1 - \alpha R - \beta R_g} J_j \right] , \quad J_i = g^{-1} \partial_i g = J_{ia} T^a , \quad (\text{A.1})$$

defines an integrable two-parameter (α, β) deformation of the principal chiral model for group G .³⁹ Here the operator R acts on the generators T^a as follows: it multiplies the generators corresponding to positive roots by i , the generators corresponding to negative roots by $-i$ and annihilates the Cartan subalgebra. The operator R_g acts on the algebra of G similarly to eq. (4.8),

$$R_g(T^a) = g^{-1}R(gT^ag^{-1})g, \quad R(T^a) \equiv \mathcal{R}^a_b T^b, \quad (\text{A.2})$$

where \mathcal{R} is the part of R in eq. (4.8) which acts on the generators of one $SU(2)$ factor and $\text{Tr}[T^a T^b] = 2\delta^{ab}$. Then

$$\mathcal{L}_K = (\eta^{ij} + \epsilon^{ij}) \Omega^{ab}(g) J_{ia} J_{jb}, \quad (\text{A.3})$$

$$\Omega^{-1} = \mathbb{1} - \alpha \mathcal{R} - \beta A(g) \mathcal{R} A^{-1}(g), \quad gT^a g^{-1} \equiv A^a_b(g) T^b. \quad (\text{A.4})$$

The deformation (A.1) may thus be interpreted as picking up a particular nonstandard (G non-invariant and in general non-symmetric) group space “metric” Ω .

For $G = SU(2)$ generated by the Pauli matrices one finds that \mathcal{R}^a_b is given by

$$\mathcal{R} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (\text{A.5})$$

Choosing the following parametrization of the group element:

$$g = g_3(\phi_1 + \phi_2) g_1(r) g_3(\phi_1 - \phi_2), \quad g_3(\phi) = \exp\left(\frac{i}{2}\phi\sigma_3\right), \quad g_1(r) = r\mathbb{1} + i\sqrt{1-r^2}\sigma_1, \quad (\text{A.6})$$

and using the explicit expressions for the matrices $A(g)$ and \mathcal{R} , one finds that the symmetric and antisymmetric parts of Ω in (A.4) are

$$\frac{1}{2}(\Omega + \Omega^T) = \mathbb{1} - H^{-1} \left[\alpha^2 \mathcal{R}^2 + \beta^2 A \mathcal{R}^2 A^{-1} - \alpha\beta(\mathcal{R} A \mathcal{R} A^{-1} + A \mathcal{R} A^{-1} \mathcal{R}) \right], \quad (\text{A.7})$$

$$\frac{1}{2}(\Omega - \Omega^T) = H^{-1}(\alpha \mathcal{R} + \beta A \mathcal{R}^2 A^{-1}), \quad H \equiv 1 + (\alpha - \beta)^2 + 4\alpha\beta r^2. \quad (\text{A.8})$$

The antisymmetric part of Ω , representing a WZ-type term in (A.3), contributes just a total derivative and thus may be ignored. The model is therefore defined by the symmetric part of Ω corresponding to the following target space metric

$$ds^2 = \frac{1}{1 + (\alpha + \beta)^2 r^2 + (\alpha - \beta)^2 (1 - r^2)} \left[\frac{dr^2}{1 - r^2} + r^2 [1 + (\alpha + \beta)^2 r^2] d\phi_1^2 + (1 - r^2) [1 + (\alpha - \beta)^2 (1 - r^2)] d\phi_2^2 + 2(\alpha^2 - \beta^2) r^2 (1 - r^2) d\phi_1 d\phi_2 \right]. \quad (\text{A.9})$$

³⁹Let us note that this integrable deformation is different from the one based on a gauged WZW type construction in [53], which is related to non-abelian T-duality. Also, the parameters α, β here should not be confused with coordinates used in the main text (cf. (3.16)).

Using the notation as in (3.6)–(3.8) (with $w \rightarrow r$, $\chi_1 \rightarrow \phi_1$, $\chi_2 \rightarrow -\phi_2$) the resulting Lagrangian may be written also in a form similar to (3.3) ($R_i^a \equiv J_i^a$)

$$\mathcal{L}_K = \frac{1}{2(1 + \alpha^2 + \beta^2 + 2\alpha\beta M)} \eta^{ij} \left[\frac{1}{2} \text{Tr}(\partial_i g \partial_j g^{-1}) + (\alpha L_i^3 + \beta R_i^3)(\alpha L_j^3 + \beta R_j^3) \right]. \quad (\text{A.10})$$

Here M and L_i^3, R_i^3 are defined as in (3.4) but now in terms of g given in eq. (A.6).

Despite the apparent dissimilarity between the Lagrangians in (A.10) and (3.3) there exists a reparametrization that relates them, i.e. a coordinate transformation that maps the metric (A.9) into the one in eq. (3.9). One is to identify the parameters and the radial coordinates as follows:

$$\begin{aligned} \alpha &= \sqrt{(d-c)(d+c+1)} = \sqrt{r(\ell+1)}, & \beta &= \sqrt{(d+c)(d-c+1)} = \sqrt{\ell(r+1)}, \\ r^2 &= \frac{1}{2} + \frac{(1+2d)z - \sqrt{[(1+d)^2 - c^2](d^2 - c^2)}(1-z^2)}{2[(1+d)^2 - c^2] - 2(d^2 - c^2)z^2}. \end{aligned} \quad (\text{A.11})$$

In the special case of $\beta = 0$ the matrix Ω in (A.4) becomes constant and the model reduces to the squashed 3-sphere one

$$\mathcal{L}_K(\beta = 0) = \frac{1}{1 + \alpha^2} \eta^{ij} \left[J_i^1 J_j^1 + J_i^2 J_j^2 + (1 + \alpha^2) J_i^3 J_j^3 \right]. \quad (\text{A.12})$$

In the equal-parameter case,

$$\alpha = \beta \equiv \frac{1}{2}\varkappa, \quad (\text{A.13})$$

the metric (A.9) becomes diagonal and is readily seen to be equivalent to the metric in (3.1). The coordinate transformation (3.15) then maps it to the symmetric case of Fateev model (3.13).

B 4-parameter integrable 3d model with a WZ term

Given that the 2-parameter Fateev model appears as a deformation of the $SO(4)/SO(3)$ coset there should exist a similar 2-parameter deformation of the $AdS_3 \times S^3$ supercoset with bosonic part, consisting of a sum of the deformed AdS_3 and S^3 spaces, being supported by some combination of RR fluxes (and dilaton). At the same time, there is also another deformation of the 3-sphere or $SU(2)$ principal chiral model (and thus also of the $AdS_3 \times S^3$ supercoset [30, 45]) corresponding to adding a WZ term with an arbitrary coefficient q (with $q = 1$ as the WZW model case). One should then expect to find an integrable 3-parameter deformation of S^3 (or AdS_3) and thus of the $AdS_3 \times S^3$ supercoset.

Indeed, a 4-parameter integrable deformation generalizing Fateev's 2-parameter deformed S^3 model to the presence of a B -field coupling was constructed by Lukyanov [26]. Below we shall review the sigma model of [26] and suggest that one of the two additional parameters is related to the WZ deformation parameter q , while the other should have a “trivial” origin as a T-duality (TsT or $O(2, 2)$ duality) transformation parameter on the two isometric directions of the model.

The action of a 3d sigma model with two translational isometries along (χ_1, χ_2) may be written as (cf. (3.9))

$$L = T \left[U(z) \partial_+ z \partial_- z + D(z) \partial_+ \chi_1 \partial_- \chi_1 + \hat{D}(z) \partial_+ \chi_2 \partial_- \chi_2 \right. \\ \left. + (C + B)(z) \partial_+ \chi_1 \partial_- \chi_2 + (C - B)(z) \partial_+ \chi_2 \partial_- \chi_1 \right], \quad (\text{B.1})$$

where C is an off-diagonal 3d metric component and B is the coefficient in the 2-form $B_2 = B(z) d\chi_1 \wedge d\chi_2$. The functions in (B.1) have the following explicit form [26]⁴⁰

$$U = \frac{m^2}{4(1 - z^2)(1 - \kappa^2 z^2)}, \\ D = R^2(1 + z) [2 + \kappa(p^2 + p^{-2}) - \kappa(2\kappa + p^2 + p^{-2})z] Q(z), \\ \hat{D} = (1 - z) [2 + \kappa(p^2 + p^{-2}) + \kappa(2\kappa + p^2 + p^{-2})z] Q(z), \\ C = \kappa(p^2 - p^{-2})R(1 - z^2) Q(z), \quad Q(z) \equiv \frac{(c + 1)(\bar{c} - 1)}{4(1 - \kappa^2)(c + z)(\bar{c} - z)}, \\ B = -\frac{m}{c + \bar{c}}(R + 1) [h(c^2 - 1)(\bar{c} - z) - \bar{h}(\bar{c}^2 - 1)(c + z)] Q(z), \quad (\text{B.2})$$

$$c^2 \equiv \frac{1 + h^2}{\kappa^2 + h^2}, \quad \bar{c}^2 \equiv \frac{1 + \bar{h}^2}{\kappa^2 + \bar{h}^2}, \quad m^2 \equiv (\kappa + p^2)(\kappa + p^{-2}), \quad R^2 \equiv \frac{(c - 1)(\bar{c} + 1)}{(c + 1)(\bar{c} - 1)}. \quad (\text{B.3})$$

The 4 independent parameters used in [26] are κ, p, h, \bar{h} , where $\kappa \in [0, 1]$ should not be confused with \varkappa in (1.1), (2.3). In the special case of

$$h = \bar{h} = 0, \quad c = \bar{c} = \kappa^{-1}, \quad R = 1, \quad Q(z) = \frac{1}{4(1 - \kappa^2 z^2)}, \quad (\text{B.4})$$

the B -field vanishes and this model reduces [26] to the Fateev model (3.9) with the following identification of parameters

$$d = \frac{1}{2}(\ell + r) = -\frac{1}{2}\kappa m^{-2}(2\kappa + p^2 + p^{-2}), \quad c = \frac{1}{2}(\ell - r) = \frac{1}{2}\kappa m^{-2}(p^2 - p^{-2}), \quad (\text{B.5})$$

$$a^2 = (1 + \ell)(1 + r) = m^{-2}, \quad b^2 = \ell r = \kappa^2 m^{-2}, \quad (\text{B.6})$$

$$\ell = -\frac{\kappa}{\kappa + p^2}, \quad r = -\frac{\kappa}{\kappa + p^{-2}}, \quad \kappa^2 = \frac{\ell r}{(\ell + 1)(r + 1)}, \quad m^2 = \frac{1}{(\ell + 1)(r + 1)}, \quad (\text{B.7})$$

with $\hat{D}(z) = D(-z)$. In the 1-parameter deformation case $\ell = r$ corresponding to $p = 1$ (see (3.12), (3.15)) we have $\varkappa^2 = 4\ell(\ell + 1)$ while $\kappa^2 = \frac{\ell^2}{(\ell + 1)^2}$, i.e.

$$\ell = r = \frac{\kappa}{1 - \kappa}, \quad \varkappa = \frac{2\sqrt{\kappa}}{1 - \kappa}. \quad (\text{B.8})$$

Thus $\kappa = \eta^2$ where η is the deformation parameter in (1.1).

⁴⁰We write the background in terms of the coordinates (χ_1, χ_2) related to (u, w) in [26] by $\chi_1 = \frac{1}{2}R^{-1}(v - w)$, $\chi_2 = \frac{1}{2}(v + w)$, $R^2 = \frac{(c-1)(\bar{c}+1)}{(c+1)(\bar{c}-1)}$. We have absorbed an overall constant in T , i.e. effectively setting g^2 of [26] to 4.

A special case with a non-zero WZ term is found for $\kappa = 0$:

$$\kappa = 0, \quad m = 1, \quad h = \bar{h}, \quad c = \bar{c} = (1 + h^{-2})^{1/2}, \quad R = 1, \quad (\text{B.9})$$

$$U = \frac{1}{4(1 - z^2)}, \quad Q = \frac{1}{4[1 + h^2(1 - z^2)]},$$

$$D(z) = \hat{D}(-z) = 2(1 + z)Q(z), \quad C = 0, \quad B = \frac{2}{\sqrt{1 + h^2}} z Q(z). \quad (\text{B.10})$$

This background represents a familiar marginal deformation (with parameter h) of the $SU(2)$ WZW model:

$$ds^2 = d\theta^2 + \frac{1}{1 + h^2 \sin^2 2\theta} (\cos^2 \theta d\chi_1^2 + \sin^2 \theta d\chi_2^2),$$

$$B_2 = \frac{1}{2\sqrt{1 + h^2}} \frac{\cos 2\theta}{1 + h^2 \sin^2 2\theta} d\chi_1 \wedge d\chi_2. \quad (\text{B.11})$$

It can be constructed by starting with the gauged WZW model for $SU(2) \times U(1)/U(1)$ or by applying an $O(2, 2)$ T-duality transformation to the $SU(2)$ WZW model (see, e.g., [54, 55]).

It is possible to make the $\kappa \rightarrow 0$ limit more non-trivial by setting [26]

$$\kappa \rightarrow 0, \quad p^2 = \frac{\kappa}{m^2 - 1} \rightarrow 0, \quad h = \bar{h} = \frac{\kappa q}{\sqrt{1 - q^2}} \rightarrow 0, \quad (\text{B.12})$$

$$c = \bar{c} \rightarrow \kappa^{-1} \sqrt{1 - q^2} \rightarrow \infty, \quad R \rightarrow 1, \quad m, q = \text{fixed}, \quad (\text{B.13})$$

$$U = \frac{1}{4} \frac{m^2}{1 - z^2}, \quad Q = \frac{1}{4}, \quad D(z) = \hat{D}(-z) = \frac{1}{4}(1 + z)[2 + (m^2 - 1)(1 - z)], \quad (\text{B.14})$$

$$C = -\frac{1}{4}(m^2 - 1)(1 - z^2), \quad B = \frac{1}{2}mqz, \quad (\text{B.15})$$

where m and q are the remaining fixed parameters related to the squashing of S^3 and the WZ term coefficient respectively. The resulting squashed S^3 metric and B -field are

$$ds^2 = m^2 d\theta^2 + \cos^2 \theta [1 + (m^2 - 1) \sin^2 \theta] d\chi_1^2 + \sin^2 \theta [1 + (m^2 - 1) \cos^2 \theta] d\chi_2^2$$

$$- (m^2 - 1) \sin^2 \theta \cos^2 \theta d\chi_1 d\chi_2, \quad z = \cos 2\theta, \quad (\text{B.16})$$

$$B_2 = \frac{1}{2}mq \cos 2\theta d\chi_1 \wedge d\chi_2. \quad (\text{B.17})$$

For $m = 1$ the corresponding Lagrangian becomes that of the $SU(2)$ principal chiral model with a WZ term with coefficient q ($q = 1$ is the case of the WZW model).

Another special case is $\kappa = 1$ (or $\varkappa = \infty$, cf. (B.7),(B.8)) when after some parameter and coordinate redefinitions [26] the background becomes equivalent to that of the marginal deformation of the euclidean $SL(2, R)$ WZW model [54, 56, 57].

Like the Fateev model [25], the above 4-parameter model is renormalizable [26],⁴¹ i.e. its form is preserved under the RG flow with only the parameters κ, h, \bar{h} and the overall scale T in

⁴¹This was checked [26] only in one-loop approximation. However, since the model has 3d target space, the corresponding curvature tensor is expressed in terms of Ricci tensor (and also the strength of B_2 is $H_{mnk} = H\epsilon_{mnk}$) and thus it is possible that there is a choice of reparametrization that demonstrates also the two-loop renormalizability.

(B.1) changing (p is not renormalized). The IR fixed point corresponds to $\kappa \rightarrow 0$ and thus to the marginal deformation of the $SU(2)$ WZW model (B.11) which becomes a Weyl-invariant sigma model when supplemented by an appropriate dilaton. The UV fixed point corresponds to $\kappa \rightarrow 1$ when the model flows to the marginal deformation of the $SL(2, R)$ WZW background, which again represents a conformal sigma model. Thus the RG flow connects the deformed S^3 and the euclidean AdS_3 or H^3 spaces just like in the case of the simple symmetric 1-parameter deformed coset model discussed in section 3 ($\varkappa = 0$ and $\varkappa = \infty$ correspond to $\kappa = 0$ and $\kappa = 1$, see (B.8)).

To find the conformal sigma model representing the string solution with the NS-NS background (B.1), (B.2), (B.3) which may correspond to a deformation of the $AdS_3 \times S^3$ supercoset with a non-zero coefficient q for the WZ term, one would need to switch on also the RR background fields (and determine the corresponding dilaton).

Finally, let us note that one of the two parameters h, \bar{h} that controls the WZ coupling in (B.1), (B.2), (B.3) may be generated by a T-duality transformation. Since T-duality formally preserves the integrability of the model (see, e.g., [3, 4, 5]) the “core” integrable 3d sigma model with two isometries may thus be characterized just by 3 parameters, that can be chosen, e.g., as the two parameters of Fateev model or κ and p and the coefficient of the WZ term. Indeed, performing the following TsT transformation:⁴² T-duality $\chi_1 \rightarrow \tilde{\chi}_1$, shift of $\chi_2 \rightarrow \chi_2 + \gamma \tilde{\chi}_1$, and reverse T-duality $\tilde{\chi}_1 \rightarrow \bar{\chi}_1$ gives a model of the same type as in (B.1) but with redefined functions D, \hat{D}, C, B containing one extra free parameter γ :

$$D' = K^{-1}D, \quad \hat{D}' = K^{-1}\hat{D}, \quad C' = K^{-1}C, \quad B' = K^{-1}[B + \gamma(B^2 + \Delta)], \quad (B.18)$$

$$K \equiv (1 + \gamma B)^2 + \gamma^2 \Delta, \quad \Delta \equiv D\hat{D} - C^2 = 4m^2 R^2 (1 - z^2)(1 - \kappa^2 z^2) Q^2. \quad (B.19)$$

The transformed functions have, in general, a different dependence on z as compared to the original one in (B.2), but this transformation is supposed to act on a special 3-parameter case to produce a 4-parameter one. In particular, starting with the 1-parameter deformation case without B -term ($p = 1, h = \bar{h} = 0$) and applying (B.18), (B.19) one gets a special case of (B.1), (B.2), (B.3) with non-zero $h = -\bar{h}$.

C The $\varkappa = i$ action from the $AdS_2 \times S^2$ supercoset

In this Appendix we include some details of the construction of the deformed supercoset action in the two $\varkappa \rightarrow i$ (or, equivalently, $\eta \rightarrow i$) limits. The details of the construction depend quite strongly on the choice of Z_4 automorphism; it turns out that a convenient one is that of [18], which identifies $\text{diag}(\sigma_1, 0)$ and $\text{diag}(0, i\sigma_1)$ as the generators of the gauge group in the $AdS_2 \times S^2$ supercoset. With this choice, the coset representative takes the form

$$g = g_B g_F, \quad g_B = \begin{pmatrix} g_A & 0_{2 \times 2} \\ 0_{2 \times 2} & g_S \end{pmatrix}, \quad g_F = \exp F, \quad (C.1)$$

$$g_A = e^{\frac{1}{2}i\tau\sigma_3} \begin{pmatrix} \cosh a & i \sinh a \\ -i \sinh a & \cosh a \end{pmatrix}, \quad g_S = e^{\frac{1}{2}i\varphi\sigma_3} \begin{pmatrix} \cos b & \sin b \\ -\sin b & \cos b \end{pmatrix}. \quad (C.2)$$

⁴²This transformation is equivalent to a non-trivial $O(2, 2)$ duality transformation [55] depending on an $O(2)$ rotation matrix with angle α such that $\gamma = -\tan \alpha$, provided one also rescales the coordinates χ_i by $\cos \alpha$ and makes a constant shift of B by γ .

Denoting by Q_i and S_i the $PSU(1,1|2)$ generators with charges $\pm i$, respectively,

$$\begin{aligned} Q_0 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & Q_1 &= \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} & Q_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & Q_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\ S_0 &= \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & S_1 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} & S_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & S_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \end{aligned} \quad (C.3)$$

we choose the matrix F defining the fermionic part of the coset representative as

$$\begin{aligned} F &= (q_0 + \eta s_0)Q_0 + (q_1 - \eta s_1)Q_1 + (q_2 + \eta s_2)Q_2 + (q_3 - \eta s_3)Q_3 \\ &+ (s_0 + \eta q_0)S_0 + (s_1 - \eta q_1)S_1 + (s_2 + \eta q_2)S_2 + (s_3 - \eta q_3)S_3. \end{aligned} \quad (C.4)$$

Here q_i and s_i are real and F obeys the reality condition outlined in Appendix C of [18]. The formal limit $\varkappa \rightarrow i$ or, equivalently $\eta \rightarrow i$, changes the reality condition obeyed by F ; such a change is hinted at [41] by the expected relation to the PR model for the $AdS_2 \times S^2$ superstring and the fact that the fundamental excitations change from magnons (in the GS theory) to solitons (in the PR theory).

With this choice of fields, the quadratic terms around the null geodesic $x^+ = t + \varphi = \mu\tau$ are diagonal and manifestly exhibit the decoupling of q_0, q_3, s_0, s_3 . This is a consequence of kappa symmetry. We will choose to fix it setting to zero the decoupled fields,

$$q_0 = 0 = q_3 = 0 = s_0 = 0 = s_3. \quad (C.5)$$

This gauge, setting to zero the diagonal entries of the upper-right and lower-left 2×2 blocks of the purely fermionic terms in the coset representative, is the analog of the $AdS_5 \times S^5$ lightcone gauge around the null geodesic.

The construction of the action from eq. (4.10) is straightforward albeit tedious; for generic η the resulting expression is quite lengthy but it simplifies in the $\eta \rightarrow i$ limit.

Taking the naive limit $\eta \rightarrow i$ (i.e. setting $\eta = i$ directly without additional rescalings) leads, after a change of coordinates

$$a \rightarrow \operatorname{arctanh}(\tan a), \quad b \rightarrow \operatorname{arctan}(\tanh b), \quad (C.6)$$

to the flat space metric. The fermionic Lagrangian then describes four free massless fermions.

As discussed in section 4.2, the limit (2.16) with $\varkappa^2 = -1 + \epsilon^2$ (i.e. $\eta = i(1 - \epsilon + \dots)$) makes contact with the PR theory for the GS string in $AdS_2 \times S^2$ [18]. In this limit, using the coset representative in eq. (C.1) and rescaling all fermions by the factor $(\eta - i)^{1/2}/(2\sqrt{\mu})$, the

Lagrangian in eq. (4.10) becomes (we use $\eta_{ij} = \text{diag}(-1, 1)$ and $\epsilon^{01} = 1$):

$$L = c_\eta (L_B + L_F) , \quad (\text{C.7})$$

$$L_B = 2\eta^{ij} \partial_i a \partial_j a + 2\eta^{ij} \partial_i b \partial_j b + \frac{\mu^2}{4} (\cos 4a - \cosh 4b) , \quad (\text{C.8})$$

$$\begin{aligned} L_F = & -q_1 \partial_+ q_1 - q_2 \partial_+ q_2 - s_1 \partial_- s_1 - s_2 \partial_- s_2 \\ & - 2 \left(\frac{\sinh 2b}{\cos 2a + \cosh 2b} \partial_- a + \frac{\sin 2a}{\cos 2a + \cosh 2b} \partial_- b \right) s_1 s_2 \\ & + 2 \left(\frac{\sinh 2b}{\cos 2a + \cosh 2b} \partial_+ a + \frac{\sin 2a}{\cos 2a + \cosh 2b} \partial_+ b \right) q_1 q_2 \\ & + \mu \frac{\sin 2a \sinh 2b}{\cos 2a + \cosh 2b} (q_1 s_2 - q_2 s_1) \\ & - \frac{\mu}{2} \frac{\cos 4a + 2 \cos 2a \cosh 2b + \cosh 4b}{\cos 2a + \cosh 2b} (q_1 s_1 + q_2 s_2) . \end{aligned} \quad (\text{C.9})$$

We notice that the connection-like terms on the second and third lines of L_F are total derivatives, $\partial_\mp \arctan(\tan a \tanh b)$, and thus they may be eliminated by opposite rotations in the planes (q_1, q_2) and (s_1, s_2) :

$$\begin{aligned} X &= \arctan(\tan a \tanh b) , \\ q_1 &\rightarrow \cos X \, q_1 + \sin X \, q_2 , & q_2 &\rightarrow -\sin X \, q_1 + \cos X \, q_2 , \\ s_1 &\rightarrow \cos X \, s_1 - \sin X \, s_2 , & s_2 &\rightarrow \sin X \, s_1 + \cos X \, s_2 . \end{aligned} \quad (\text{C.10})$$

The resulting fermionic Lagrangian is

$$\begin{aligned} L_F = & -q_1 \partial_+ q_1 - q_2 \partial_+ q_2 - s_1 \partial_- s_1 - s_2 \partial_- s_2 \\ & + \mu \left[\cos 2a \cosh 2b (s_1 q_1 + s_2 q_2) + \sin 2a \sinh 2b (q_1 s_2 - q_2 s_1) \right] . \end{aligned} \quad (\text{C.11})$$

The complete light-cone gauge-fixed deformed supercoset Lagrangian to quadratic order in fermions can then be mapped to the Lagrangian of the PR model for the $AdS_2 \times S^2$ superstring [18] by a double-Wick rotation and identifying the fields as

$$a = \frac{1}{2}\varphi , \quad b = \frac{1}{2}\phi , \quad q_1 = \nu , \quad q_2 = \rho , \quad s_1 = \beta , \quad s_2 = \gamma , \quad (\text{C.12})$$

and accounting for the difference in the definition of partial derivatives ∂_\pm in [18] (see footnote 30).

We expect that a similar derivation, showing equivalence with the corresponding PR model, should be possible also for the $AdS_3 \times S^3$ deformed supercoset in the limit $\eta \rightarrow i$.

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